MATH 511A, HOMEWORK 4

Let σ be an automorphism of $\mathbb{F}$, and let V, W be $\mathbb{F}$-vector spaces. The goal of this homework assignment is to prove the following theorem.

Theorem. Let $\langle \ , \ \rangle : V \times W \to \mathbb{F}$ be a perfect σ-sesquilinear pairing of finite-dimensional vector spaces. If $\ell \in V^*$, there exists $w \in W$ such that $\ell(v) = \langle v, w \rangle$ for all $v \in V$.

Recall that a function $f : V \to W$ is called σ-*linear* if $f(v_1 + v_2) = f(v_1) + f(v_2)$ and $f(cv) = \sigma(c)f(v)$. Let the set of σ-linear maps $V \to W$ be denoted $\mathrm{Hom}_\sigma(V, W)$.

1.

(a) Check that $\mathrm{Hom}_\sigma(V, W)$ is actually an $\mathbb{F}$-vector space, where $(f_1 + f_2)(v) = f_1(v) + f_2(v)$ and $(a \cdot f)(v) = af(v)$.
(b) If $\{v_i\}$ are a basis for V and $\{w_i\}$ are arbitrary elements of W, show that there exists a unique element $f \in \mathrm{Hom}_\sigma(V, W)$ with $f(v_i) = w_i$ for all i.
(c) Define a map $T : \mathrm{Hom}_\sigma(V, W) \to \mathrm{Hom}(V, W)$ as follows: if $f \in \mathrm{Hom}_\sigma(V, W)$, then $T(f)$ is the unique linear map sending $v_i \mapsto f(v_i)$ for all i. Prove that T is an isomorphism of vector spaces. In particular deduce that $\dim \mathrm{Hom}_\sigma(V, W) = \dim \mathrm{Hom}(V, W)$.

2. Suppose that $f \in \mathrm{Hom}_\sigma(V, W)$.

(a) If f is injective, prove that $\dim(V) \leq \dim(W)$.
(b) If f is surjective, prove that $\dim(V) \geq \dim(W)$.
(c) If f is a bijection, prove that $\dim(V) = \dim(W)$.

3. Suppose $\langle \ , \ \rangle : V \times W \to \mathbb{F}$ is a pairing.

(a) We obtain a map $\phi : W \to V^*$ sending w to the map $v \mapsto \langle v, w \rangle$. Show that ϕ is σ-linear.
(b) Similarly we obtain a map $\psi : V \to \mathrm{Hom}_\sigma(W, \mathbb{F})$ sending v to the map $w \mapsto \langle v, w \rangle$. Show that ψ is linear.

4.

(a) If $\langle \ , \ \rangle : V \times W \to \mathbb{F}$ is a perfect pairing and either V or W is finite-dimensional, prove that $\dim(V) = \dim(W)$.
(b) Give an example to show that if V, W are infinite-dimensional and $\langle \ , \ \rangle : V \times W \to \mathbb{F}$ is a perfect pairing, then it need not be the case that $\dim(V) = \dim(W)$.

5. Prove the theorem.