MATH 511A, HOMEWORK 7

1. If H and K are finite subgroups of a group G and if $\#H$, $\#K$ are relatively prime, prove that $H \cap K = 1$. (Here 1 denotes the trivial subgroup of G.)

2. Prove that A_4 and D_{12} are non-isomorphic groups of order 12.

3.

(i) If $f : G \to H$ is a homomorphism and $x \in G$ has order k, prove that $f(x) \in H$ has order dividing k.
(ii) If $\#G, \#H$ are relatively prime and $f : G \to H$ is a homomorphism, prove that $f(x) = 1$ for all $x \in G$.

4. Prove that if $\sigma \in S_n$ then σ and σ^{-1} are conjugate. Give an example of a group G and an element $g \in G$ such that g and g^{-1} are not conjugate.

5. The *center* of a group G, denoted $Z(G)$, is the set $\{x \in G : xy = yx \text{ for all } y \in G\}$.

(i) Prove that $Z(\mathrm{GL}_2(\mathbb{F}))$ is the set of scalar matrices.
(ii) If G is a group and $G/Z(G)$ is cyclic, prove that G is abelian.

6. Let Q be the quaternion group of order 8. (This is the group H_8 defined on page 127 of the textbook.) Prove that $Q/Z(Q) \cong V$. (So the requirement that $G/Z(G)$ be cyclic rather than merely abelian in problem 5(ii) is essential.) Prove also that Q has no subgroup isomorphic to V, so that $Q/Z(Q)$ is not isomorphism to a subgroup of Q.

7. If G is a group, an isomorphism $f : G \to G$ is called an *automorphism* of G. The set $\mathrm{Aut}(G)$ of all automorphisms of G is a group under composition.

(i) If $g \in G$, prove that the map $\gamma_g : G \to G$ defined by $\gamma_g(x) = gxg^{-1}$ is an automorphism of G.
(ii) Prove that the function $\Gamma : G \to \mathrm{Aut}(G)$ with $\Gamma(g) = \gamma_g$ is a homomorphism.
(iii) Prove that $\ker(\Gamma) = Z(G)$.
(iv) Define $\mathrm{Inn}(G) = \mathrm{im}(\Gamma)$, the group of *inner automorphisms* of G. Prove that $\mathrm{Inn}(G) \lhd \mathrm{Aut}(G)$.

8. Prove that $\mathrm{Aut}(V)$ and $\mathrm{Aut}(S_3)$ are both isomorphic to S_3. Prove that $\mathrm{Aut}(\mathbb{Z})$ is a cyclic group of order 2.

9. Let G be a finite group and $K \lhd G$ a normal subgroup. If $\#K$ and $[G : K]$ are relatively prime, prove that K is the unique subgroup of G of order $\#K$.

10. Let G be a finite abelian group of order mn with m, n relatively prime, and let the group operation in G be written additively (i.e. the operation is denoted $+$). For any integer d, define

$$G_d = \{g \in G \ : \ \mathrm{ord}(g) \mid d\} \, .$$

(i) Prove that G_d is a subgroup of G, and that $G_m \cap G_n = \{0\}$.
(ii) Prove that $G = G_m + G_n = \{g + h \ : \ g \in G_m \text{ and } h \in G_n\}$.
(iii) Prove that $G \cong G_m \times G_n$.