MATH 511B, SOLUTIONS 1

1.

(a) If T is invertible then it is invertible as a map of sets, and so it is one-to-one and onto.

Conversely, if T is one-to-one and onto then there exists a map of sets $T^{-1} : W \to V$ such that $T^{-1} \circ T$ is the identity on V and $T \circ T^{-1}$ is the identity on W, and we must show that T^{-1} is actually a linear transformation. Since T is injective we have

$$T^{-1}(w_1) + T^{-1}(w_2) = T^{-1}(w_1 + w_2)$$

if and only if

$$T(T^{-1}(w_1) + T^{-1}(w_2)) = T(T^{-1}(w_1 + w_2)) \,.$$

Since T is linear and $T \circ T^{-1}$ is the identity, the left-hand side is $T(T^{-1}(w_1)) + T(T^{-1}(w_2)) = w_1 + w_2$; similarly the right-hand side is $w_1 + w_2$. Since the two sides are equal, it follows that T^{-1} is additive. The proof that T^{-1} preserves scalar multiplication is similar.

(b) If $\{v_i\}$ is a basis and $w \in W$, use the fact that T is surjective to choose v such that $Tv = w$. Write $v = \sum a_i v_i$. Then $w = \sum a_i T(v_i)$, so the set $\{T(v_i)\}$ spans W. On the other hand, if $\sum a_i T(v_i) = 0$ then $T(\sum a_i v_i) = 0$; but T is injective, and so we have $\sum a_i v_i = 0$. Since the set $\{v_i\}$ is linearly independent, we conclude that $a_i = 0$ for all i and so the set $\{T(v_i)\}$ is linearly independent. This proves that if $\{v_i\}$ is a basis then $\{T(v_i)\}$ is a basis.

Conversely suppose $\{T(v_i)\}$ is a basis of W. By part (a) we know that T^{-1} is invertible, and so by the previous paragraph $\{v_i\} = \{T^{-1}(T(v_i))\}$ is a basis of V.

Finally, if $T : V \to W$ is invertible, this shows that a basis for V has the same cardinality as a basis of W, hence $\dim(V) = \dim(W)$.

2. Consider the three types of row operations:

(1) Swap row i and row j,
(2) Multiply row i by a nonzero constant c,
(3) Add c times row i to row j.

The effect of each of these row operations on a column of A, considered as a vector in $\mathbb{F}^k$, is as follows:

(1) Swap entry i and entry j,
(2) Multiply entry i by a nonzero constant c,
(3) Add c times entry i to entry j.

Note that each of these operations is an invertible linear transformation $\mathbb{F}^k$.

Let A' be the result of performing a single row operation on A, and T the corresponding invertible linear transformation of $\mathbb{F}^k$, so that if $v_1, \dots, v_n$ are the columns of A then $T(v_1), \dots, T(v_n)$ are the columns of A'. Suppose $v_{i_1}, \dots, v_{i_r}$ are a basis for $\langle v_i \;:\; i = 1, \dots, n\rangle$. Since T is invertible, by #1(b) we know that $T(v_{i_1}), \dots, T(v_{i_r})$ are still a basis for $\langle T(v_i) \;:\; i = 1, \dots, n\rangle$. That is, if the $i_1, \dots, i_r$th columns are a basis for the space spanned by the columns of A, then after performing a row operation the $i_1, \dots, i_r$th columns are a basis for A'.

In particular, the dimension of the space spanned by the columns of A' is equal to the dimension of the space spanned by the columns of A. Since one row operation does not change this dimension, repeatedly applying row operations does not change the dimension either, and therefore the dimension of the space spanned by the columns of A_0 is equal to the dimension of the space spanned by the columns of A.

3. We consider each row operation in turn:

(1) Swapping row i and row j is achieved by multiplying on the left by the matrix $E_{i,j}$ with all entries zero except $e_{ij} = e_{ji} = 1$, $e_{kk} = 1$ if $k \neq i, j$. This matrix is its own inverse.
(2) Multiplying row i by a nonzero constant c is achieved by multiplying on the left by the matrix $E_{i,c}$ with all entries zero except $e_{ii} = c$ and $e_{kk} = 1$ if $k \neq i$. The inverse of this matrix is $E_{i,c^{-1}}$.
(3) Adding c times row i to row j is achieved by multiplying on the left by the matrix $E_{i,j,c}$ with all entries zero except $e_{kk} = 1$ for all k and $e_{ji} = c$. The inverse of this matrix is $E_{i,j,-c}$.

The second sentence follows because a product of invertible matrices is invertible.

4. Since $A_0 = BA$ with B invertible, each row of A_0 is a *nontrivial* linear combination of the rows of A. (The invertible matrix B cannot have any row consisting entirely of zeros.) Therefore if A_0 has a row consisting entirely of zeros, the rows of A are linearly dependent. On the other hand, if the rows of A are linearly dependent, suppose that row k that occurs in a nontrivial linear dependence. Then there is a sequence of row operations of type "add c times row i to row k" such that the resulting matrix has a row consisting entirely of zeros. Swap this row to the last row, and row reduce the remaining rows to reduced row echelon form. The result must be A_0, the unique reduced row echelon form for A, and so A_0 contains a row of zeros.

5. The idea is to give a diagonalization-type construction of an element $v \in \mathbb{F}^{\mathbb{N}}$ that does not lie in the span of $\langle v_1, v_2, \dots\rangle$, or equivalently, does not lie in $\langle v_1, \dots, v_n\rangle$ for any n.

For each n we define a linear transformation $T_n : \mathbb{F}^{\mathbb{N}} \to \mathbb{F}^{n+1}$ by projection onto the coordinates $\binom{n+1}{2}$ through $\binom{n+2}{2} - 1$. (That is, T_1 projects onto the first two coordinates, T_2 projects onto the next three coordinates, and so forth.)

Note that $\langle v_1, \dots, v_n\rangle$ has dimension at most n, and so $T_n(\langle v_1, \dots, v_n\rangle) \subset \mathbb{F}^{n+1}$ is a proper subspace of $\mathbb{F}^{n+1}$. In particular for each n we can choose $v^{(n)} \in \mathbb{F}^{n+1}$

that is not in this image. Let v be the vector such that for each n, the coordinates $\binom{n+1}{2}$ through $\binom{n+2}{2} - 1$ of v are given by the entries of $v^{(n)}$. Then $T_n(v) = v^{(n)} \notin T_n(\langle v_1, \ldots, v_n \rangle)$ for all n, and we conclude that $v \notin \langle v_1, \ldots, v_n \rangle$ for all n.