MATH 511A, SOLUTIONS TO HOMEWORK 11

1.

(a) To check that an element lies in the center of G, it suffices to check that it commutes with all the generators of G: since the centralizer $C_G(x)$ is a subgroup of G, if it contains a set of generators of G then it is equal to G.

It is clear that the z_i's commute with one another, since they are disjoint from one another. Similarly z_0 is disjoint from y_2, y_3; z_1 is disjoint from y_1, y_3; z_2 is disjoint from y_1, y_2; z_3 is disjoint from y_0, y_3.

It remains to check that z_0 commutes with y_0, y_1; z_1 with y_0, y_2; z_2 with y_0, y_3; and z_3 with y_1, y_2. These are all checked by combining the following three facts: (i) disjoint cycles commute; (ii) any cycle commutes with itself; (iii) the Klein four-group is commutative, so any $(2,2)$-cycle commutes with any other $(2,2)$-cycle permuting the same four numbers.

(b) Since z_0, z_1, z_2, z_3 commute and have order two, the subgroup they generate (which is certainly contained in the center) consists of all elements of the form

$$z_0^{a_0} z_1^{a_1} z_2^{a_2} z_3^{a_3} \tag{$\star$}$$

with $a_i = 0, 1$ for $i = 0, 1, 2, 3$. Since each z_i is disjoint from the others, if at least one a_i is nonzero then the product $(\star)$ is not the identity. It follows that all 16 products of the form $(\star)$ are distinct.

2. Initially it appears that there are 16 commutators to check. However, it's trivial that $[y_i, y_i] = 1$ for all i. On the other hand one checks from the definition that $[y_i, y_j] = [y_j, y_i]^{-1}$; since the elements of $\{1, z_0, \ldots, z_3\}$ are their own inverses, if $[y_j, y_i]$ is one of these elements then so is $[y_i, y_j]$.

Now y_3 commutes with y_1, y_2 since it is disjoint from them. Since $(5,7)(9,11)$ is disjoint from y_1 and $(13,15)$ is disjoint from y_0, the commutator of y_0, y_1 is the same as the commutator of $(1,3)$ and $(1,2)(3,4)$, and this is

$$((1,3)(1,2)(3,4))^2 = (3,4,1,2)^2 = (1,3)(2,4) = z_0\,.$$

Similarly the commutator $[y_0, y_2]$ is the commutator of $(5,7)$ with $(5,6)(7,8)$, which is $(5,7)(6,8) = z_1$; the commutator $[y_0, y_3]$ is the commutator of $(9,11)$ with $(9,10)(11,12)$, which is $(9,11)(10,12) = z_2$; the commutator $[y_1, y_2]$ is the commutator of $(13,15)$ with $(13,14)(15,16)$, which is $(13,15)(14,16) = z_3$.

By the last two paragraphs, we have $Z = \langle z_0, z_1, z_2, z_3 \rangle \subset G'$. Note that $Z \triangleleft G$, since Z is contained in the center of G, so to prove that $Z = G'$ it suffices to prove that G/Z is abelian. To see this, note that G/Z is generated by the cosets $y_i Z$ for $i = 0, \ldots, 3$, and that $[y_i Z, y_j Z] = [y_i, y_j]Z = Z$ since $[y_i, y_j] \in Z$ for all i, j.

3. Since $y_i Z$ (for $i = 0, \ldots, 3$) has order at most two, and since $y_i Z$ commutes with $y_j Z$, it follows that every element of G/Z has the form $y_0^{\epsilon_0} y_1^{\epsilon_1} y_2^{\epsilon_2} y_3^{\epsilon_3} Z$ with

$\epsilon_i \in \{0,1\}$ for all i. Hence G/Z has order at most 16. Moreover, two of these 16 elements are equal if and only if one of them is nontrivially equal to to the identity, if and only if some $y_0^{\epsilon_0} y_1^{\epsilon_1} y_2^{\epsilon_2} y_3^{\epsilon_3}$ with at least one ϵ_i equal to 1 lies in Z.

4.

(a) We make repeated use of the following claim: if $x, y \in G$ and $[x,y] = z \in Z(G)$, then $x^a y^b = z^{ab} y^b x^a$. This is true for all a, b but we shall prove it (and apply it) only for $a, b \geq 0$. The statement is trivial when $a, b = 0$, and it is true by definition for $a = b = 1$. Now we check the statement when $a = 1$, by induction on b. We have already checked this for $b = 0, 1$, while if $b > 0$ we have

$$\begin{aligned} xy^b &= (xy^{b-1})y \\ &= z^{b-1} y^{b-1} xy \\ &= z^{b-1} y^{b-1} zyx \\ &= z^b y^b x\,, \end{aligned}$$

completing the induction. (In the second line we have used the induction hypothesis, and in the fourth line we have used the hypothesis that $z \in Z(G)$.) Now we argue that the statement is true for arbitrary a, b, by induction on a (with the cases $a = 0, 1$ known). Then

$$\begin{aligned} x^a y^b &= x(x^{a-1} y^b) \\ &= x(z^{(a-1)b} y^b x^{a-1} \\ &= z^{(a-1)b} x y^b x^{a-1} \\ &= z^{(a-1)b} z^b y^b x x^{a-1} \\ &= z^{ab} y^b x^a \end{aligned}$$

as desired, where in the second line we have used the induction hypothesis, and in the fourth line we have made use of the case $a = 1$.

Now by repeated use of our lemma and by the computations in #2 we see that

$$(y_0^{\epsilon_0} y_1^{\epsilon_1} y_2^{\epsilon_2} y_3^{\epsilon_3}) \cdot (y_0^{\delta_0} y_1^{\delta_1} y_2^{\delta_2} y_3^{\delta_3})$$

is equal to

$$z_2^{\delta_0 \epsilon_3} y_0^{\epsilon_0} y_1^{\epsilon_1} y_2^{\epsilon_2} y_0^{\delta_0} y_3^{\epsilon_3} y_1^{\delta_1} y_2^{\delta_2} y_3^{\delta_3}$$

is equal to

$$z_1^{\delta_0 \epsilon_2} z_2^{\delta_0 \epsilon_3} y_0^{\epsilon_0} y_1^{\epsilon_1} y_0^{\delta_0} y_2^{\epsilon_2} y_3^{\epsilon_3} y_1^{\delta_1} y_2^{\delta_2} y_3^{\delta_3}$$

is equal to

$$z_0^{\delta_0 \epsilon_1} z_1^{\delta_0 \epsilon_2} z_2^{\delta_0 \epsilon_3} y_0^{\epsilon_0 + \delta_0} y_1^{\epsilon_1} y_2^{\epsilon_2} y_3^{\epsilon_3} y_1^{\delta_1} y_2^{\delta_2} y_3^{\delta_3}$$

is equal to

$$z_0^{\delta_0 \epsilon_1} z_1^{\delta_0 \epsilon_2} z_2^{\delta_0 \epsilon_3} y_0^{\epsilon_0 + \delta_0} y_1^{\epsilon_1} y_2^{\epsilon_2} y_1^{\delta_1} y_2^{\delta_2} y_3^{\delta_3 + \epsilon_3}$$

is equal to

$$z_0^{\delta_0 \epsilon_1} z_1^{\delta_0 \epsilon_2} z_2^{\delta_0 \epsilon_3} z_3^{\delta_1 \epsilon_2} y_0^{\delta_0 + \epsilon_0} y_1^{\delta_1 + \epsilon_1} y_2^{\delta_2 + \epsilon_2} y_3^{\delta_3 + \epsilon_3}$$

as desired.

(b) By another application of the formula in part (a) we have that

$$(y_0^{\delta_0} y_1^{\delta_1} y_2^{\delta_2} y_3^{\delta_3}) \cdot (y_0^{\epsilon_0} y_1^{\epsilon_1} y_2^{\epsilon_2} y_3^{\epsilon_3})$$

is equal to

$$z_0^{\delta_1\epsilon_0} z_1^{\delta_2\epsilon_0} z_2^{\delta_3\epsilon_0} z_3^{\delta_2\epsilon_0} y_0^{\delta_0+\epsilon_0} y_1^{\delta_1+\epsilon_1} y_2^{\delta_2+\epsilon_2} y_3^{\delta_3+\epsilon_3} .$$

Since the desired commutator is $(y_0^{\epsilon_0} y_1^{\epsilon_1} y_2^{\epsilon_2} y_3^{\epsilon_3}) \cdot (y_0^{\delta_0} y_1^{\delta_1} y_2^{\delta_2} y_3^{\delta_3})$ times the inverse of $(y_0^{\delta_0} y_1^{\delta_1} y_2^{\delta_2} y_3^{\delta_3}) \cdot (y_0^{\epsilon_0} y_1^{\epsilon_1} y_2^{\epsilon_2} y_3^{\epsilon_3})$, the computation follows.

(c)

If $y_0^{\epsilon_0} y_1^{\epsilon_1} y_2^{\epsilon_2} y_3^{\epsilon_3}$ lies in $Z(G)$, then it commutes with all elements $y_0^{\delta_0} y_1^{\delta_1} y_2^{\delta_2} y_3^{\delta_3}$. By #1(b) this is possible only if the exponents $z_0, \ldots, z_3$ in the formula of #4(b) are zero for all choices of δ_i, which is possible only if all the ϵ_i are equal to zero.

5.

(a) The formula in #4(a) implies that the set of elements in G of the form $z_0^{a_0} z_1^{a_1} z_2^{a_2} z_3^{a_3} y_0^{\epsilon_0} y_1^{\epsilon_1} y_2^{\epsilon_2} y_3^{\epsilon_3}$ is closed under multiplication, hence (since this set contains generators for G, and since G is finite) every element in G has this form. If this element lies in the center, then so does $y_0^{\epsilon_0} y_1^{\epsilon_1} y_2^{\epsilon_2} y_3^{\epsilon_3}$, and by #4(c) we have $\epsilon_i = 0$ for all i. This proves $Z(G) \subset Z$, hence $Z(G) = Z$. By #2 we conclude that $Z(G) = G'$ and $\#Z(G) = 16$.

(b) By #4(c) we have seen that no element of the form $y_0^{\epsilon_0} y_1^{\epsilon_1} y_2^{\epsilon_2} y_3^{\epsilon_3}$ lies in Z, so by #3 the order of $G/Z = G/Z(G)$ is 16.

(c) Parts (a) and (b) together imply that #G = 256.

6. If $z \in Z(G)$, note that $[zx, y] = [x, zy] = [x, y]$ for any $x, y \in G$. By the argument in #5(a), every element of G has the form $z y_0^{\epsilon_0} y_1^{\epsilon_1} y_2^{\epsilon_2} y_3^{\epsilon_3}$ with $z \in Z(G)$. If $z, z' \in Z(G)$, note that $[zx, z'y] = [x, y]$ for any $x, y \in G$. It follows that every commutator of elements of G is equal to a commutator of the form

$$[y_0^{\epsilon_0} y_1^{\epsilon_1} y_2^{\epsilon_2} y_3^{\epsilon_3}, y_0^{\delta_0} y_1^{\delta_1} y_2^{\delta_2} y_3^{\delta_3}]$$

and is therefore equal to

$$z_0^{\delta_0\epsilon_1-\delta_1\epsilon_0} z_1^{\delta_0\epsilon_2-\delta_2\epsilon_0} z_2^{\delta_0\epsilon_3-\delta_3\epsilon_0} z_3^{\delta_1\epsilon_2-\delta_2\epsilon_1} .$$

If $\delta_0 = \epsilon_0 = 0$ then this commutator is a power of z_3, hence is not equal to $z_2 z_3$. If $\epsilon_0 = \delta_0 = 1$, then the exponents of z_0, z_1 are zero only if $\epsilon_1 = \delta_1$ and $\epsilon_2 = \delta_2$, in which case the exponent of z_3 is zero; so again this commutator is not equal to $z_2 z_3$. If $\delta_0 = 0$ while $\epsilon_0 = 1$, then the exponents of z_0, z_1 are zero only if $\delta_1 = \delta_2 = 0$, and again the exponent of z_3 is zero and the commutator is not $z_2 z_3$. The case $\delta_0 = 1, \epsilon_0 = 0$ is the same. We have checked all possibilities for δ_0, ϵ_0, and conclude that $z_2 z_3$ is not a commutator.

(In fact, all other fifteen elements of G' *are* commutators!)