MATH 511A, SOLUTIONS TO HOMEWORK 12

1. For the first equality, note that $(1, k) \in C_G(H)$ if and only if

$$(h, 1) = (1, k)(h, 1)(1, k^{-1}) = (\phi(k)(h), 1))$$

for all h. This is the case if and only if $\phi(k) = 1$, i.e., if and only if $k \in \ker(\phi)$.

For the second equality, note that if $(h, 1) \in N_G(K)$ then $(h, 1)(1, k)(h, 1)^{-1} = (1, k')$ for some k'. But we compute

$$(h, 1)(1, k)(h, 1)^{-1} = (h, k)(h^{-1}, 1) = (h\phi(k)(h^{-1}), k)\,.$$

This is equal to $(1, k')$ only if $k = k'$, which implies $H \cap N_G(K) \subset H \cap C_G(K)$. The reverse inclusion is trivial.

2. If $H \rtimes_\phi K$ is abelian then

$$\ker(\phi) = C_G(H) \cap K = G \cap K = K$$

by problem #1. Hence ϕ is trivial. The converse is evident.

3.

(a) Let G be a group of order 75. A Sylow 5-subgroup P_5 of G has index 3, so is normal. Let P_3 be a Sylow 3-subgroup of G. Since $P_5P_3 = G$, $P_5 \cap P_3 = \{e\}$, and $P_5 \lhd G$, we conclude that $G \cong P_5 \rtimes_\phi P_3$ for some $\phi : P_3 \to \mathrm{Aut}(P_5)$. We have two cases: $P_5 \cong \mathbb{Z}/25\mathbb{Z}$ and $P_5 \cong (\mathbb{Z}/5\mathbb{Z})^2$.

In the first case $\mathrm{Aut}(P_5) \cong (\mathbb{Z}/25\mathbb{Z})^\times \cong \mathbb{Z}/20\mathbb{Z}$ so the only map $P_3 \to \mathrm{Aut}(P_5)$ is the trivial one. Hence $G \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/25\mathbb{Z} \cong \mathbb{Z}/75\mathbb{Z}$ in this case.

In the second case, $\mathrm{Aut}(P_5) \cong \mathrm{GL}_2(\mathbb{Z}/5\mathbb{Z})$. The order of $\#GL_2(\mathbb{Z}/5\mathbb{Z})$ is $(5^2 - 1)(5^2 - 5) = 480$. Since $3 \parallel 480$, a subgroup of $\mathrm{GL}_2(\mathbb{Z}/3\mathbb{Z})$ of order 3 is a Sylow 3-subgroup; therefore all subgroups of order 3 are conjugate. It follows from Corollary 3 in the semidirect products handout that all nontrivial maps $P_3 \to \mathrm{Aut}(P_5)$ yield isomorphic semidirect products. By Problem #2, if $\phi : P_3 \to \mathrm{Aut}(P_5)$ is nontrivial then G is nonabelian. So in this case we obtain two nonisomorphic groups of order 75: the abelian group $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$, and a nonabelian group $(\mathbb{Z}/5\mathbb{Z})^2 \rtimes_\phi \mathbb{Z}/3\mathbb{Z}$ for any nontrivial map $\phi : \mathbb{Z}/3\mathbb{Z} \to \mathrm{GL}_2(\mathbb{Z}/5\mathbb{Z})$.

We conclude that there are a total of 3 groups of order 75.

(b) Both abelian groups of order 75 certainly have a subgroup of order 15. What about the nonabelian group $G = (\mathbb{Z}/5\mathbb{Z})^2 \rtimes_\phi \mathbb{Z}/3\mathbb{Z}$? Note that since $3 \nmid 5 - 1$, every group of order 15 is cyclic.

Solution 1: In G, the number n_3 of Sylow 3-subgroups must divide 25, so is 1, 5, or 25. But $n_3 = 5$ is ruled out because $5 \not\equiv 1 \pmod 3$. Also, if we had $n_3 = 1$, then $P_3 \lhd G$ as well as $P_5 \lhd G$, and this would imply that $G \cong P_3 \times P_5$ is abelian, which it isn't. So $n_3 = 25$. But then G contains $25 \cdot 2 = 50$ elements of order 3, in addition to the 25 elements in P_5. This accounts for all the elements in G, so none have order 15.

Solution 2: Argue as before, and note that a group has a subgroup of order 15 if and only if there is an element of order 3 that commutes with an element of order 5. Since any two subgroups of order 3 in G are conjugate (they are Sylow 3-subgroups), one element of order 3 commutes with an element of order 5 if and only if every element of order 3 commutes with some element of order 5 (not always the same element of order 5, of course).

Since $n_3 = 25$, we have $[G : N_G(P_3)] = 25$, and so $\#N_G(P_3) = 3$. This shows that the normalizer in G of P_3 doesn't contain any elements of order 3, so there aren't any elements of order 5 that commute with the elements of P_3.

Solution 3: Argue as before that G has a subgroup of order 15 if and only if every element of order 3 commutes with some element of order 5. Since $G = (\mathbb{Z}/5\mathbb{Z})^2 \rtimes_\phi \mathbb{Z}/3\mathbb{Z}$, there is a unique Sylow 5-subgroup, which contains all the elements of order 5. Since the structure of G doesn't depend on the choice of ϕ, let's take ϕ to be the map which sends a generator $x \in \mathbb{Z}/3\mathbb{Z}$ to $A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, which you can check is an element of order 3. This means that in G, the action of the element x by conjugation on $(\mathbb{Z}/5\mathbb{Z})^2$ is given by the matrix A. The element x commutes with an element y of order 5 if and only if $xyx^{-1} = y$, if and only if y is fixed by A. But A doesn't fix any elements in $(\mathbb{Z}/5\mathbb{Z})$ — for instance, because $A - I$ is invertible. So x does not commute with any elements of order 5, and G does not have a subgroup of order 15.

In fact, it isn't necessary to choose a particular map ϕ. The point is that no element A of order 3 in $\mathrm{GL}_2(\mathbb{Z}/5\mathbb{Z})$ has 1 as an eigenvalue: the polynomial $x^3 - 1$ factors into irreducibles over $\mathbb{Z}/5\mathbb{Z}$ as $(x-1)(x^2+x+1)$, and so the only possibilities for the minimal polynomial of A are $x - 1$ and $x^2 + x + 1$. In the former case $A = I$, and in the latter case 1 is not an eigenvalue (since 1 is not a root of $x^2 + x + 1$). Although this is the most complicated of the solutions we've given to this problem, it's the one that is most amenable to generalization.

4.

(a) If $\#G = 28$ we must have $n_7 = 1$, so $G \cong \mathbb{Z}/7\mathbb{Z} \rtimes_\phi P_2$ for some map $\phi : P_2 \to \mathrm{Aut}(\mathbb{Z}/7\mathbb{Z}) \cong \mathbb{Z}/6\mathbb{Z}$, where $P_2 \in \mathrm{Syl}_7(G)$ has order 4. There are two possibilities: $P_2 \cong \mathbb{Z}/4\mathbb{Z}$ and $P_2 \cong V$. If ϕ is trivial, we obtain the abelian groups $\mathbb{Z}/28\mathbb{Z}$ and $\mathbb{Z}/14\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Suppose that ϕ is nontrivial.

There is a unique nontrivial map $\mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z}$. So, if $P_2 \cong \mathbb{Z}/4\mathbb{Z}$, there is one nonabelian semidirect product $\mathbb{Z}/7\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$.

Any nontrivial map $V \to \mathbb{Z}/6\mathbb{Z}$ has kernel isomorphic to $\mathbb{Z}/2\mathbb{Z}$, and for each subgroup of order 2 in V there is a unique map $V \to \mathbb{Z}/6\mathbb{Z}$ with that kernel. Since any two subgroups of order 2 in V can be interchanged by an automorphism of V, it follows from Lemma 2 on the semidirect products handout that all these maps yield isomorphic semidirect products. So, there is one nonabelian semidirect product $\mathbb{Z}/7Z \rtimes V$.

(b) As in #6, we have $n_5 = 1$, so $G \cong \mathbb{Z}/5\mathbb{Z} \rtimes_\phi P_2$ for $\phi : P_2 \to \mathrm{Aut}(\mathbb{Z}/5\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}$, with two possibilities for P_2. If ϕ is trivial we obtain two abelian groups. Suppose ϕ is nontrivial.

By exactly the same argument as in #5 there is only one nonabelian semidirect product $\mathbb{Z}/5\mathbb{Z} \rtimes V$. However, there are several nontrivial maps $\mathbb{Z}/4\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/5\mathbb{Z}) \cong \mathbb{Z}/4\mathbb{Z}$. There are two isomorphisms $\mathbb{Z}/4\mathbb{Z} \cong \mathrm{Aut}(\mathbb{Z}/5\mathbb{Z})$; they are interchanged by the automorphism $x \mapsto -x$ of $\mathbb{Z}/4Z$, so by Lemma 2 of the semidirect products handout these two maps yield just one nonabelian semidirect product $\mathbb{Z}/5\mathbb{Z} \rtimes_{\phi_1} \mathbb{Z}/4\mathbb{Z}$ with $\ker(\phi_1) = 1$. There is also a one map $\mathbb{Z}/4\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/5\mathbb{Z})$ with kernel of order 2, so we have a semidirect product $\mathbb{Z}/5\mathbb{Z} \rtimes_{\phi_2} \mathbb{Z}/4\mathbb{Z}$ with $\# \ker(\phi_2) = 2$. By Corollary 7 of the semidirect products handout, $\mathbb{Z}/5\mathbb{Z} \rtimes_{\phi_1} \mathbb{Z}/4\mathbb{Z} \not\cong \mathbb{Z}/5\mathbb{Z} \rtimes_{\phi_2} \mathbb{Z}/4\mathbb{Z}$.

So, there are a total of 5 groups of order 20.

5. The wreath product $\mathbb{Z}/p\mathbb{Z} \wr \mathbb{Z}/p\mathbb{Z}$ is a semidirect product $(\mathbb{Z}/p\mathbb{Z})^p \rtimes (\mathbb{Z}/p\mathbb{Z})$, so its order is $p^p \cdot p = p^{p+1}$. On the other hand, the largest power of p dividing $\#S_{p^2} = (p^2)!$ is exactly p^{p+1} (since p^2 is divisible by p multiples of p, one of which is a multiple of p^2). Therefore a Sylow p-subgroup of S_{p^2} has order p^{p+1}. Since all Sylow p-subgroups of S_{p^2} are conjugate, hence isomorphic, it suffices to find a subgroup of S_{p^2} that is isomorphic to $\mathbb{Z}/p\mathbb{Z} \wr \mathbb{Z}/p\mathbb{Z}$.

For $i = 1, \ldots, p$, let $\sigma_i \in S_{p^2}$ be the p-cycle

$$((1 + p(i-1))\ (2 + p(i-1))\ \cdots\ (p + p(i-1)))\,,$$

so, e.g., $\sigma_1 = (1\ 2\ \cdots\ p)$ and $\sigma_p = ((p^2 - p + 1)\ \cdots\ p^2)$. Then σ_i and σ_j are disjoint if $i \neq j$, and the subgroup

$$\langle \sigma_1 \rangle \times \cdots \times \langle \sigma_p \rangle \subset S_{p^2}$$

is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^n$.

Let τ be the permutation

$$(1\ p+1\ \cdots\ (p^2 - p + 1))(2\ p+2\ \cdots\ (p^2 - p + 2)) \cdots (p\ 2p\ \cdots\ p^2)\,.$$

Then $\tau \sigma_i \tau^{-1} = \sigma_{i+1}$ for all i. Then

$$\tau((\sigma_1^{a_1}, \ldots, \sigma_p^{a_p}))\tau^{-1} = (\sigma_1^{a_p}, \sigma_2^{a_1}, \ldots, \sigma_p^{a_{p-1}})\,.$$

So the action of τ on $\langle \sigma_1 \rangle \times \cdots \times \langle \sigma_p \rangle$ is precisely the wreath product action, and the subgroup of S_{p^2} generated by τ and $\sigma_1, \ldots, \sigma_p$ is isomorphic to $\mathbb{Z}/p\mathbb{Z} \wr \mathbb{Z}/p\mathbb{Z}$.