MATH 511B, SOLUTIONS 3

1. Let $v_1, \dots, v_n$ be a basis for V (over $\mathbb{C}$). We claim that $v_1, \dots, v_n, i \cdot v_1, \dots, i \cdot v_n$ are a basis for $V_{\mathbb{R}}$. Indeed, suppose $v \in V_{\mathbb{R}}$. Regarded as an element of V, we may write

$$v = z_1 v_1 + \cdots + z_n v_n$$

with each $z_j \in \mathbb{C}$. Write $z_j = x_j + iy_j$ with $x_j, y_j \in \mathbb{R}$. Then $z_j v_j = (x_j + iy_j)v_j = x_j v_j + y_j(i \cdot v_j)$, and therefore

$$v = x_1 v_1 + \cdots + x_n v_n + y_1(i \cdot v_1) + \cdots + y_n(i \cdot v_n)\,.$$

Thus the vectors $v_1, \dots, v_n, i \cdot v_1, \dots, i \cdot v_n$ span $V_{\mathbb{R}}$.

On the other hand, suppose

$$0 = x_1 v_1 + \cdots + x_n v_n + y_1(i \cdot v_1) + \cdots + y_n(i \cdot v_n)\,.$$

As above, this means

$$0 = z_1 v_1 + \cdots + z_n v_n$$

where $z_j = x_j + iy_j$, and since $v_1, \dots, v_n$ are linearly independent we have $z_j = 0$ for all j. This implies $x_j, y_j = 0$ for all j. Hence the vectors $v_1, \dots, v_n, i \cdot v_1, \dots, i \cdot v_n$ are linearly independent; combined with the previous paragraph, we see that they are a basis of $V_{\mathbb{R}}$, and therefore $\dim_{\mathbb{R}}(V_{\mathbb{R}}) = 2 \dim_{\mathbb{C}}(V)$.

2. In order for the sum to be direct, we need $S \cap T = 0$, that is, we need the equations $A^t = A$ and $A^t = -A$ to imply $A = 0$. If $A^t = A$ and $A^t = -A$ then $A = -A$, so $2A = 0$. If the characteristic of $\mathbb{F}$ is not equal to 2, this implies $A = 0$, as desired. On the other hand, if the characteristic of $\mathbb{F}$ is 2, then we can draw no such conclusion; in fact note in this case that A is always equal to $-A$, and so $S = T$.

So it is certainly necessary that the characteristic of $\mathbb{F}$ is not 2. Let us suppose this is the case. Then we have already shown that $S \cap T = 0$, and it remains to show that $S + T = M_n(\mathbb{F})$. If $A \in M_n(\mathbb{F})$, set $B = \frac{1}{2}(A + A^t)$ and $C = \frac{1}{2}(A - A^t)$. (Note that the fraction makes sense because $2 \neq 0$ in $\mathbb{F}$.) One checks easily that $B \in S$, $C \in T$, and $A = B + C$, as desired.

Therefore it is necessary and sufficient that the characteristic of $\mathbb{F}$ is not 2, or equivalently, that 2 is invertible in $\mathbb{F}$.

3. In fact we will prove that if $V_1, \dots, V_n \subsetneq V$ and $\cup_i V_i = V$ then $\#\mathbb{F} \leq n - 1$. This inequality is best-possible, because if $\#\mathbb{F} = q$ then a two-dimensional vector space over $\mathbb{F}$ can be decomposed into a union of $q + 1$ lines.

Suppose for the sake of contradiction that $\#\mathbb{F} > n - 1$. Let k be the smallest positive integer such that $V_1 \cup \cdots \cup V_k = V$, so that $1 < k \leq n$. Since $V_k \neq V$ we can choose $w \notin V_k$. Let v be an arbitrary element of V_k. Then $w + \alpha v \notin V_k$ for any $\alpha \in \mathbb{F}$, hence $w + \alpha v$ lies in V_i for some $i < k$. Since $\#\mathbb{F} > n - 1 \geq k - 1$, by the pigeonhole principle there is some $i < k$ so that V_i contains two elements $w + \alpha v$ and $w + \beta v$ with $\alpha \neq \beta$; then $v \in V_i$ as well. This proves that $V_1 \cup \cdots \cup V_{k-1}$ contains V_k, hence $V_1 \cup \cdots \cup V_{k-1} = V$. This contradicts the choice of k, completing the proof.

4. Let $f = \sum_{k=0}^{n} a_k x^k$ be a polynomial in $\mathbb{F}[x]$, so that $Df = \sum_{k=0}^{n} k a_k x^{k-1}$. Then $Df = 0$ if and only if $k a_k = 0$ in $\mathbb{F}$ for all k; or equivalently, if and only if whenever $a_k \neq 0$ we have $k = 0$ *in the field* $\mathbb{F}$.

If $\mathbb{F}$ has characteristic 0, then $k = 0$ in $\mathbb{F}$ only when $k = 0$. In this case $Df = 0$ if and only if the only nonzero term in f is the constant term.

If $\mathbb{F}$ has characteristic 0, then $k = 0$ if $\mathbb{F}$ if and only if k is divisible by p. In this case $Df = 0$ if and only if the only nonzero terms in f are of degree divisible by p, that is, if and only if $f(x) = g(x^p)$ for some polynomial $g(x) \in \mathbb{F}[x]$. This completes the determination of $\ker(D)$.

As for the image of D, suppose first that $\mathbb{F}$ has characteristic 0. Set $g = \sum_{k=0}^{n} a_k (k+1)^{-1} x^{k+1}$. One checks easily that $Dg = f$, and so D is surjective.

On the other hand, if $\mathbb{F}$ has characteristic p, then $(k+1)^{-1}$ does not exist if $k+1$ is divisible by p. In fact, note from the formula $Df = \sum k a_k x^{k-1}$ that an element in the image of D never includes a nonzero term of degree $pj - 1$ for any j. On the other hand, if f has no nonzero term of degree $pj - 1$ for any j then we can set

$$g = \sum_{k \not\equiv -1 \pmod{p}} a_k (k+1)^{-1} x^{k+1}$$

and $Dg = f$. Hence the image of D is the space of polynomials with no nonzero terms of degree congruent to $-1 \pmod{p}$.