MATH 511B, SOLUTIONS 4

1.

(a) One must check that the given operations are well-defined. For instance, we verify that $(a \cdot f)$ is σ-linear: we have

$$(a \cdot f)(cv) = af(cv) = a\sigma(c)f(v) = \sigma(c)(af(v)) = \sigma(c)(a \cdot f)(v)$$

as desired. The rest of the check is similarly straightforward.

(b) If $v \in V$, write $v = \sum c_i v_i$; if there exists a σ-linear map f with the necessary property, then we must have $f(v) = \sum \sigma(c_i) f(v_i) = \sum \sigma(c_i) w_i$. Since the function f defined via this formula is easily checked to be σ-linear and to satisfy $f(v_i) = w_i$, the desired function exists and is unique.

(c) First we check that T is linear. If $f, g \in \mathrm{Hom}_\sigma(V, W)$ then $T(f + g)$ sends $v_i \mapsto (f + g)(v_i) = f(v_i) + g(v_i)$, so $T(f + g) = T(f) + T(g)$. If $a \in \mathbb{F}$ then $T(af)$ sends $v_i \mapsto (af)(v_i) = a \cdot f(v_i)$, so $T(af) = aT(f)$.

To check that T is surjective, suppose $g \in \mathrm{Hom}(V, W)$. By part (b) there is a map $f \in \mathrm{Hom}_\sigma(V, W)$ with $f(v_i) = g(v_i)$. Then $T(f)(v_i) = f(v_i) = g(v_i)$ for all i, hence $T(f) = g$.

To check that T is injective, suppose $T(f) = 0$. Then $f(v_i) = 0$ for all i, and by uniqueness in part (b) we have $f = 0$.

2.

(a) Let $\{v_i\}$ be a basis of V. Suppose that $\sum c_i f(v_i) = 0$ in W. We write $c_i = \sigma(d_i)$ with $d_i \in \mathbb{F}$, so that

$$0 = \sum c_i f(v_i) = \sum f(d_i v_i) = f(\sum d_i v_i)\,.$$

By injectivity of f we have $\sum d_i v_i$, and since the vectors $\{v_i\}$ are linearly independent we have $d_i = 0$ for all i. Hence $c_i = 0$ for all i as well, and the vectors $\{f(v_i)\}$ are linearly independent in W. Therefore $\dim(V) \le \dim(W)$.

(b) Again let $\{v_i\}$ be a basis of V. Note that $f(v)$ lies in the linear span of the vectors $\{f(v_i)\}$; since f is surjective we conclude that the vectors $\{f(v_i)\}$ span W, which gives the desired inequality.

(c) This is immediate from (a) and (b) together.

3.

(a) First, $\phi(w_1 + w_2)$ is the map $v \mapsto \langle v, w_1 + w_2 \rangle = \langle v, w_1 \rangle + \langle v, w_2 \rangle$, so $\phi(w_1 + w_2) = \phi(w_1) + \phi(w_2)$. Second, $\phi(cw)$ is the map $v \mapsto \langle v, cw \rangle = \sigma(c)\langle v, w \rangle$, so $\phi(cw) = \sigma(c)\phi(w)$. This completes the check.

(b) Note that the map $w \mapsto \langle v, w \rangle$ is indeed an element in $\mathrm{Hom}_\sigma(W, \mathbb{F})$. As in part (a) we have $\psi(v_1 + v_2) = \psi(v_1) + \psi(v_2)$. Now $\psi(cv)$ is the map $w \mapsto \langle cv, w \rangle = c\langle v, w \rangle = c(\psi(v)(w))$, so $\psi(cv) = c\psi(v)$ and ψ is linear.

4.

(a) By definition the pairing $\langle\ ,\ \rangle \to \mathbb{F}$ is perfect if and only if the maps ϕ, ψ in problem #3 are injective. From parts 3(a) and 2(a) we conclude $\dim(W) \leq \dim(V^*)$. From parts 3(b) and 1(c) we conclude $\dim(V) \leq \dim(W^*)$. From these inequalities we see that if one of V, W is finite dimensional, then so is the other. Moreover in this case we have $\dim(V) = \dim(V^*)$ and $\dim(W) = \dim(W^*)$, and along with the above inequalities this implies $\dim(V) = \dim(W)$.

(b) If $\sigma = 1$, the pairing $V \times V^* \to \mathbb{F}$ with $\langle v, \ell\rangle = \ell(v)$ is always a perfect pairing, and if V is infinite-dimensional then $\dim(V)$ and $\dim(V^*)$ are not equal.

5. The theorem is equivalent to the statement that under the hypotheses of the theorem, the map ϕ in 3(a) is surjective. By problem 4, we have $\dim(V) = \dim(W)$. Let n be this dimension. The map $\phi : W \to V^*$ from problem 3(a) is a σ-linear injection of one n-dimensional vector space into another. If $w_1, \ldots, w_n$ are a basis of W then the proof of 2(a) shows that $\phi(w_1), \ldots, \phi(w_n)$ are linearly independent in V^*, hence a basis of V^*, so any element $\ell \in V^*$ may be written $\ell = \sum_i c_i\phi(w_i)$ for elements $c_i \in \mathbb{F}$. Since σ is an automorphism, we can choose d_i so that $c_i = \sigma(d_i)$; then

$$\ell = \sum_i c_i\phi(w_i) = \sum_i \sigma(d_i)\phi(w_i) = \phi(\sum_i d_i w_i)\,.$$

Hence ϕ is surjective, as desired.