MATH 445, HOMEWORK 1

1. Let A, B be events in a probability space S. Prove that

$$P(A \cap B) + P(A \cup B) = P(A) + P(B)\,.$$

Conclude that if A, B are *mutually exclusive*, i.e., if $A \cap B = \emptyset$, then $P(A \cup B) = P(A) + P(B)$.

2. You intercept a bigram "....KL..." of ciphertext that has been encrypted with a substitution cipher.

(a) What is the probability that K stands for the letter I?
(b) If you learn that K does stand for I, what is the probability that L stands for one of N,S,T?

3. Revisit the "medical test" scenario described in class. Suppose the doctor orders the same test a second time, and suppose that the second test is independent of the first. The test is positive again. Now what is the probability that you have the disease?

4. Let S be the probability space consisting of all possible outcomes of n flips of a fair coin. Prove (working directly from the definition of independence) that the outcome of the nth flip is independent of the outcome of the preceding flips. (That is, if H_i is the random variable whose value is 1 if the ith flip is heads and 0 if the ith flip is tails, prove that H_n is independent of $H_1, \ldots, H_{n-1}$.)

5. The *binomial coefficient* $\binom{n}{m}$ is the number of different m-element subsets in an n-element set. (Since this is the number of different ways of choosing a collection of m objects from a collection of n objects, one reads the symbol $\binom{n}{m}$ aloud as "n choose m".) The formula $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ should be familiar to you — if it isn't, please look it up and make sure you understand why it is true.

(a) Suppose $0 < p < 1$. Use Stirling's formula to prove the estimate

$$\binom{n}{pn} \sim \frac{1}{\sqrt{2\pi p(1-p)}} \left(\frac{1}{p^p(1-p)^{1-p}}\right)^n n^{-1/2}\,.$$

(b) As an application of part (a), show that

$$\binom{2n}{n} 2^{-2n} \sim \frac{1}{\sqrt{\pi n}}\,.$$