MATH 511B, PROBLEMS 10

Here are some more Galois theory problems.

1. Let ζ_n be a primitive nth root of unity; set $K = \mathbb{Q}(\zeta_n)$ and $K^+ = \mathbb{Q}(\zeta_n + \zeta_n^{-1})$. Prove that $[K : K^+] = 2$; show moreover that the image of any map $K^+ \hookrightarrow \mathbb{C}$ lies in $\mathbb{R}$. For this reason, K^+ is called the *maximal totally real subfield* of K.

2. Suppose $\pm\alpha, \pm\beta$ are the roots of $f(x) = x^4 + ax + b \in \mathbb{Q}[x]$. Prove that f is irreducible if and only if α^2 and $\alpha \pm \beta$ are not elements of $\mathbb{Q}$. In that case, let G be the Galois group of $f(x)$, and show:

(1) $G \cong V_4$ if and only if b is square in $\mathbb{Q}$;
(2) $G \cong \mathbb{Z}/4\mathbb{Z}$ if and only if $b(a^2 - 4b)$ is square in $\mathbb{Q}$;
(3) $G \cong D_8$ otherwise.

3. Let f be a polynomial of degree n and f' its derivative. Prove that the discriminant of f is given by the formula

$$\operatorname{disc}(f) = (-1)^{n(n-1)/2} \prod_{i=1}^{n} f'(\alpha_i)$$

where $\alpha_1, \dots, \alpha_n$ are the roots of f.

4. Use the previous problem to compute $\operatorname{disc}(\Phi_n)$ where Φ_n is the nth cyclotomic polynomial. If p is prime, deduce that the unique quadratic field inside $\mathbb{Q}(\zeta_p)$ is $\mathbb{Q}(\sqrt{(-1)^{(p-1)/2}p})$.

5. (Artin-Schreier extensions) Let F be a field of characteristic p and let K be a cyclic extension of F of degree p. Prove that $K = F(\alpha)$ where α is a root of the polynomial $x^p - x - a$ for some $a \in F$.

6. Let $D \in \mathbb{Z}$ be a squarefree integer and let $a \in \mathbb{Q}$ be a nonzero rational number. If $\mathbb{Q}(\sqrt{a\sqrt{D}})/\mathbb{Q}$ is Galois, prove that $D = -1$ and $\operatorname{Gal}(\mathbb{Q}(\sqrt{a\sqrt{D}})/\mathbb{Q}) \cong V_4$ or $\mathbb{Z}/2\mathbb{Z}$. For which values of a is the Galois group V_4, and for which is it $\mathbb{Z}/2\mathbb{Z}$?