MATH 511B, HOMEWORK 3

1. If R is an integral domain with fraction field K, prove that the polynomial ring $R[x]$ is an integral domain with fraction field $K(x)$.

2. Suppose that $f : R \to R'$ is a map of rings. If P' is a prime ideal in R', prove that $f^{-1}(P')$ is a prime ideal in R. In the other direction, if P is a prime ideal in R, give examples to show (i) that $f(P)$ need not even be an ideal in R, but also (ii) the ideal *generated* by $f(P)$ need not be prime.

3. If R, S are rings, determine the prime ideals of $R \times S$ (in terms of the prime ideals of R and S).

Definition. If I, J are ideals in R, let $I + J$ be the ideal $\{i + j \,:\, i \in I,\, j \in J\}$, and let IJ be the ideal $\{\sum_k i_k j_k \,|\, i_k \in I,\, j_k \in J\}$.

4.

(i) If I, J are ideals in R, prove that the natural map $R \to R/I \times R/J$ is an isomorphism if and only if $I + J = R$ and $I \cap J = 0$.

(ii) Generalize (i) as follows. If $I_1, \ldots, I_n$ are ideals in R, prove that $R \to R/I_1 \times \cdots \times R/I_n$ is surjective if and only if $I_i + I_j = R$ for all $i \neq j$ (and an isomorphism if also $I_1 \cap \cdots \cap I_n = 0$).

5. If $I + J = R$, prove that $IJ = I \cap J$.