MATH 511B, HOMEWORK 7

1. Determine a splitting field and its degree over $\mathbb{Q}$ for $x^4 + 2$.

2. Determine a splitting field and its degree over $\mathbb{Q}$ for $x^4 + x^2 + 1$.

3. Determine a splitting field and its degree over $\mathbb{Q}$ for $x^6 - 4$.

Definition: A finite extension K of F is said to be a splitting field over F if there exists $f(x) \in F[x]$ such that K is the splitting field of f over F.

4. Let K be a finite extension of F. Prove that K is a splitting field over F if and only if every irreducible polynomial in $F[x]$ that has a root in K splits completely in $K[x]$.

5. Let K_1, K_2 be finite extensions of F contained in the field K, and assume that both are splitting fields over F.

(i) Prove that their composite K_1K_2 is a splitting field over F.
(ii) Prove that $K_1 \cap K_2$ is a splitting field over F.

6. Let L/K be an algebraic extension. If R is a sub*ring* of L containing K, show that R is actually a field. Give a counterexample if L/K is not algebraic.

7. Let $\mathbb{F}_p(t,u)$ be the field of rational functions in the two variables t, u over $\mathbb{F}_p$. Prove that $[\mathbb{F}_p(t,u) : \mathbb{F}_p(t^p, u^p)] = p^2$, and that there exist infinitely many fields K lying strictly between $\mathbb{F}_p(t,u)$ and $\mathbb{F}_p(t^p, u^p)$.

8. If $f, g \in K[x]$, verify the formula $D(fg) = fD(g) + gD(f)$.

9. Suppose K is a perfect field and f is a separable polynomial over K. If F/K is any extension, prove that f is a separable polynomial over F.

10. If $\mathbb{F}_p$ is the finite field of order p and $a \in \mathbb{F}_p^\times$, prove that $x^p - x - a$ is an irreducible and separable polynomial in $\mathbb{F}_p[x]$.