MATH 511A, HOMEWORK 8

1. Let K be a field and $K(x)$ the field of rational functions in one variable over K. If $y \in K(x) \setminus K$, prove that the extension $K(x)/K(y)$ is algebraic.

2. Let $f, g \in \mathbb{F}_2[x]$ be the polynomials $x^3 + x + 1$ and $x^3 + x^2 + 1$ respectively. Prove that f and g are irreducible. The fields $\mathbb{F}_2[x]/(f)$ and $\mathbb{F}_2[x]/(g)$ both have order 8, and therefore must be isomorphic. Give an explicit isomorphism between them.

3. Let $a_1, \ldots, a_k$ be integers and let $K = \mathbb{Q}(\sqrt{a_1}, \ldots, \sqrt{a_k})$. Prove that $[K : \mathbb{Q}]$ is a power of 2. If $[K : \mathbb{Q}] = 2^k$, write down a basis for K as a vector space K over $\mathbb{Q}$ and describe the group of automorphisms of K fixing $\mathbb{Q}$.

4. Let L/K be an extension of degree 2.

(a) If the characteristic of K is not equal to 2, prove that $L = K(\sqrt{D})$ for some $D \in K$.
(b) If the characteristic of K is equal to 2, prove that $L = K(\sqrt{D})$ for some $D \in K$ if and only if L/K is *inseparable*.

5. Let $f \in K[x]$ be a polynomial of degree n and let L/K be a splitting field of f. Prove that $\mathrm{Aut}(L/K)$ can be identified with a subgroup of S_n. If L/K is separable, deduce that $[L : K]$ divides $n!$.

6. In the last part of the previous problem, what if L/K is inseparable?

7. Let L be the splitting field over $\mathbb{Q}$ of the polynomial $x^p - 2$. Determine the group $\mathrm{Aut}(L/\mathbb{Q})$. Check that $\mathrm{Hom}_{\mathbb{Q}}(L, \overline{\mathbb{Q}}) = \mathrm{Aut}(L/\mathbb{Q})$.