Calculus*

Math 294A: Problem Solving Seminar – Vera Furst

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Hard calculus problems are common on the Putnam Exam. The problems below require (more or less) the techniques of Calculus I–III and not those of a real analysis course. That said, the key tools at your disposal are:

- The fundamental theorem of calculus (1 and 2).
- The intermediate value theorem.
- The mean value theorem (differential and integral).
- L'Hopital's Rule.

Example 1. Suppose f is a twice continuously differentiable function such that $|f(x)| \leq 1$ for all x and $(f(0))^2 + (f'(0))^2 = 4$. Prove that there exists a real number x_0 such that $f(x_0) + f''(x_0) = 0$.

Example 2. Show that the improper integral

$$\int_0^\infty \sin x \sin x^2 \; dx$$

converges.

Example 3. Let A be a positive real number. What are the possible values for $\sum_{i=0}^{\infty} x_i^2$, given that $\{x_i\}_{i=0}^{\infty}$ is a sequence of positive numbers that satisfy $\sum_{i=0}^{\infty} x_i = A$?

Example 4. Find the maximum value of the function

$$F(y) = \int_0^y \sqrt{x^4 + (y - y^2)^2} \; dx.$$

Problem 1. Find all possible functions $f : \mathbb{R} \to \mathbb{R}$ with continuous derivative f' such that

$$(f(x))^2 = \int_0^x \left[(f(t))^2 + (f'(t))^2\right] \, dt + 1990.$$

Problem 2. Evaluate the integral

$$\int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}} \; dx.$$

Problem 3. A clock's minute hand has length 4, and its hour hand has length 3. What is the distance between the tips at the moment when this distance is increasing most rapidly?

*These problems (or some version of them) all appear either on previous Putnam exams or in the book *Problem-Solving Through Problems* by L.C. Larson.

Problem 4. For what positive real numbers α, β does the integral

$$\int_{\beta}^{\infty} \sqrt{\sqrt{x+\alpha}-\sqrt{x}} - \sqrt{\sqrt{x}-\sqrt{x-\beta}}\, dx$$

converge?

Problem 5. Find

$$\lim_{x\to\infty} x \int_0^x e^{t^2-x^2}\, dt.$$

Problem 6. Evaluate

$$\lim_{n\to\infty} \frac{1}{n^4} \prod_{i=1}^{2n} (n^2+i^2)^{1/n}.$$

Problem 7.

(a) Suppose that $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are continuous and that $g(x) \geq 0$ for all x in $[a,b]$. Prove that there exists a number c in $[a,b]$ such that

$$\int_a^b f(x)g(x)\, dx = f(c) \int_a^b g(x)\, dx.$$

(b) Suppose that $f : [a,b] \to \mathbb{R}$ is increasing (and therefore integrable), and $g : [a,b] \to \mathbb{R}$ is integrable and satisfies $g(x) \geq 0$ for all x in $[a,b]$. Prove that there exists a number c in $[a,b]$ such that

$$\int_a^b f(x)g(x)\, dx = f(a) \int_a^c g(x)\, dx + f(b) \int_c^b g(x)\, dx.$$

Problem 8. Let $C(\alpha)$ denote the coefficient of x^{1992} in the power series for $(1-x)^{\alpha}$. Find

$$\int_0^1 C(-y-1) \sum_{k=1}^{1992} \frac{1}{y+k}\, dy.$$

Problem 9. Suppose that f is differentiable, and that $f'(x)$ is strictly increasing for $x \geq 0$. If $f(0) = 0$, prove that $f(x)/x$ is strictly increasing for $x > 0$.

Problem 10. Let f be a differentiable function on $[0.1]$ with $f(0) = 0$ and $f(1) = 1$.

(a) For each positive integer n, show that there exist distinct points $x_1, x_2, \ldots, x_n$ in $[0,1]$ such that

$$\sum_{i=1}^{n} \frac{1}{f'(x_i)} = n.$$

(b) For each positive integer n and arbitrary positive numbers $k_1, k_2, \ldots, k_n$, show that there exist distinct points $x_1, x_2, \ldots, x_n$ such that

$$\sum_{i=1}^{n} \frac{k_i}{f'(x_i)} = \sum_{i=1}^{n} k_i.$$

Problem 11. In the (x,y)-plane, let R be the set of points inside and on a convex polygon, and let $D(x,y)$ be the distance from (x,y) to the nearest point of R. Show that there exist constants a, b, and c, independent of R, such that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-D(x,y)}\, dx\, dy = a + bP + cA,$$

where P and A are the perimeter and area of R, respectively. Find the values of a, b, and c.

Problem 12. Sum the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$.