

Combinatorics and Sums

Math 294A: Problem Solving Seminar

Combinatorics is the mathematics of counting objects in various ways. One of the basic notions in combinatorics is the k th binomial coefficient of degree n , or the number of ways to choose k objects out of n objects (a quantity often called “ n choose k ”), and is equal to

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}, \quad \text{where } n \geq k \geq 0.$$

Note that we have the identities

$$\binom{n}{k} = \binom{n}{n-k} \quad \text{and} \quad \binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1} \quad \text{if } n \geq k \geq 1.$$

The following is often useful in finding sums involving binomial coefficients, and is of course why we call $\binom{n}{k}$ a binomial coefficient in the first place.

The Binomial Theorem. Let x and y be variables, and $n \geq 0$ an integer. Then

$$(x + y)^n = \sum_{i=0}^n \binom{n}{i} x^{n-i} y^i.$$

Not all of the problems below will use binomial coefficients, but they all deal with some type of enumeration.

Example 1. Show that for any integer $n \geq 1$, $\sum_{i=1}^n i \binom{n}{i} = n2^{n-1}$.

Example 2. Let $n > 1$. How many ways can the set $\{1, 2, \dots, n\}$ be written as the union of k nonempty disjoint subsets, each consisting of consecutive integers?

Example 3. Give a combinatorial proof that if $m, n \geq 1$ and $1 \leq k \leq n$, then

$$\sum_{i=0}^k \binom{n}{i} \binom{m}{k-i} = \binom{m+n}{k}.$$

Example 4. Consider all $2^n - 1$ nonempty subsets of $\{1, 2, \dots, n\}$. For each of these subsets, we find the product of the reciprocals of each of its elements. Find the sum of all of these products.

Problem 1. a) Prove that for $n \geq 0$,

$$\sum_{i=0}^n (-1)^i \binom{n}{i} = 0.$$

b) Define μ as a function on the positive integers by $\mu(1) = 1$, or if $n > 1$ has prime factorization $n = p_1^{e_1} \cdots p_k^{e_k}$ by

$$\mu(n) = \begin{cases} (-1)^k & \text{if } e_i = 1 \text{ for each } i, \\ 0 & \text{otherwise.} \end{cases}$$

Using part a), prove that for any positive integer n ,

$$\sum_{d|n} \mu(d) = 0.$$

Problem 2. Prove that the product of n consecutive integers is always divisible by $n!$.

Problem 3. Prove that if $n \geq 1$, then

$$\sum_{i=0}^{n-1} \frac{(-1)^i}{i+1} \binom{n}{i} = \frac{(-1)^{n+1} + 1}{n+1}.$$

Problem 4. How many subsets of $\{1, 2, \dots, n\}$ have no two successive numbers?

Problem 5. What is the probability of an odd number of sixes turning up in a random toss of n fair dice? (Hint: Consider $\frac{1}{2}((x+y)^n - (x-y)^n)$.)

Problem 6. In the Lotto, six numbers are chosen from $\{1, 2, \dots, 49\}$. How many of these six element subsets have at least a pair of consecutive numbers?

Problem 7. Let $0 < a_1 < a_2 < \dots < a_n$ be real numbers, and let $e_i = \pm 1$. Prove that $\sum_{i=1}^n e_i a_i$ takes at least $\binom{n+1}{2}$ distinct values as the e_i range over the 2^n possible combinations of signs.

Problem 8. Prove that for each positive integer n ,

$$\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}.$$

(Hint: Use the Binomial Theorem.)

Problem 9. How many ways can you select two disjoint subsets from a set consisting of n elements? We select the two subsets as an *unordered* pair, so we do not distinguish which order the two subsets are selected.

Problem 10. Let $1 \leq r \leq n$ and consider all subsets of r elements of the set $\{1, 2, \dots, n\}$. Each of these subsets has a smallest element. Let $F(n, r)$ denote the arithmetic mean of these smallest elements. Prove that $F(n, r) = \frac{n+1}{r+1}$.

Problem 11. Prove that

$$\sum_{k=0}^n \left[\frac{n-2k}{n} \binom{n}{k} \right]^2 = \frac{2}{n} \binom{2n-2}{n-1}.$$

Problem 12. Prove that

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \frac{1}{k+m+1} = \sum_{k=0}^m (-1)^k \binom{m}{k} \frac{1}{k+n+1}.$$