Number Theory

Problem 1: Show there are infinitely many primes.

Problem 2: Show the equation $x^2 + y^2 = 3z^2$ has no solutions in integers x, y, z other than (x, y, z)

Euler's Theorem: If a is relatively prime to n then $a^{\phi(n)} \equiv 1 \bmod n$.

Problem 3: Find the last two digits of 3^{1243}.

Problem 4: Show that if p is prime then each prime factor of $2^p - 1$ is at least as big as p.

1. Show there are infinitely many primes of the form $6n - 1$.
2. Prove that $21n + 4$ is relatively prime to $14n + 3$ for all n.
3. How many zeroes end the number 1000!?
4. For how many k is $\binom{100}{k}$ odd?
5. Show that the product of any n consecutive integers is divisible by $n!$.
6. Find the last two digits of 3^{1234}.
7. If $2n + 1$ and $3n + 1$ are both perfect squares, show that n is divisible by 40.
8. Find all solutions to the equation $n_1^4 + n_2^4 + ... + n_{14}^4 = 1599$.
9. Show that any integer N has a multiple whose only digits are 1's and 0's. Show that if N is relatively prime to 10 then N divides a number of the form 11111...11.

10. Show that $n^2 + 1$ is not divisible by 7 for any integer n.
11. Find all positive d such that d divides both $n^2 + 1$ and $(n+1)^2 + 1$ for some n.
12. Prove that the equation $x^2 + y^2 + z^2 = 32816887$ has no solutions.
13. Find all solutions to the equation $x^y = y^x$ with $x \neq y$ and prove that there are no others.
14. Prove the sequence $11, 111, 1111, ...$ contains no perfect squares.
15. Prove that 17 divides $2x + 3y$ if and only if 17 divides $9x + 5y$.
16. Prove that the equation $x^2 + 3xy - 2y^2 = 122$ has no integer solutions.
17. Show that it is impossible to tile a 25 by 25 checkerboard with 2×2 squares and 3×3 squares.
18. Find all positive integer solutions to the equation $3^x + 4^y = 5^z$.
19. Do there exist Fibonacci numbers with arbitrarily large numbers of terminal zeroes?
20. Prove the only rational solution of the equation $x^3 + 3y^3 + 9z^3 - 9xyz = 0$ is $x = y = z = 0$.