MATH 413/513 SUPPLEMENTARY NOTES: DUALITY

Our goal in this handout is to explain (using the dimension theory of finite-dimensional vector spaces) why the row rank and the column rank of a matrix are equal. This will turn out to be a consequence of *duality.*

Definition 1. Let V be a vector space over a field $\mathbf{F}$. Its *dual space* is

$$V^* := \mathcal{L}(V, \mathbf{F}),$$

the $\mathbf{F}$-vector space of linear maps from V to $\mathbf{F}$.

Suppose that V is finite-dimensional. Since $\dim_{\mathbf{F}} \mathbf{F} = 1$, we have

$$\dim_{\mathbf{F}} V^* = \dim_{\mathbf{F}} \mathcal{L}(V, \mathbf{F}) = (\dim_{\mathbf{F}} V)(\dim_{\mathbf{F}} \mathbf{F}) = \dim_{\mathbf{F}} V.$$

That is, if V is finite-dimensional then V^* is also finite-dimensional and has the same dimension as V.

The crucial observation is that it's possible to define not only the dual of a *vector space*, but also the dual of a *linear map* between vector spaces. Here's how this goes. Suppose that $T : V \to W$ is a linear map. If $f \in W^*$, so that f is a map $W \to \mathbf{F}$, then the composition $f \circ T$ is a linear map $V \to \mathbf{F}$; that is, $f \circ T$ is an element of V^*. Therefore we may define *the dual of* T to be the map

$$T^* : W^* \to V^*$$

sending $f \mapsto f \circ T$. It easy to check that T^* is linear. Notice that if T is a map from V to W, then T^* is a map between the duals but in the other direction.

Example 2. Consider $V = \mathbf{F}^n$, equipped with the standard basis $e_1, \ldots, e_n$. Then $V^* = \mathcal{L}(\mathbf{F}^n, \mathbf{F})$ is an n-dimensional space. Let e_i^* be the element of V^* which sends $e_i \mapsto 1$ and $e_j \mapsto 0$ for $j \neq i$. Then one checks that $e_1^*, \ldots, e_n^* \in V^*$ are linearly independent, so they are a basis of V^*, called the *standard dual basis.*

Example 3. Suppose that $T : \mathbf{F}^n \to \mathbf{F}^m$, and suppose that the matrix of T (with respect to the standard basis on $\mathbf{F}^n$ and $\mathbf{F}^m$) is A. Let us compute the matrix of $T^* : (\mathbf{F}^m)^* \to (\mathbf{F}^n)^*$ with respect to the standard dual bases on $(\mathbf{F}^m)^*$ and $(\mathbf{F}^n)^*$. (This gets a little bit complicated!)

To avoid confusion, let $e_1, \ldots, e_m$ be the standard basis on $\mathbf{F}^m$, and let $\epsilon_1, \ldots, \epsilon_n$ be the standard basis on $\mathbf{F}^n$; and let $e_1^*, \ldots, e_m^*$ denote the standard dual basis on $(\mathbf{F}^m)^*$ and let $\epsilon_1^*, \ldots, \epsilon_n^*$ denote the standard dual basis on $(\mathbf{F}^n)^*$. To compute the matrix of T^* we must expand $T^*(e_j^*)$ in terms of $\epsilon_1^*, \ldots, \epsilon_n^*$. Now $T^*(e_j^*) : \mathbf{F}^n \to \mathbf{F}$, so we can evaluate

$$T^*(e_j^*)(\epsilon_i) = e_j^*(T\epsilon_i) = e_j^*(a_{1i}e_1 + \cdots + a_{mi}e_m) = a_{ji}.$$

It follows that

$$T^*(e_j^*) = a_{j1}\epsilon_1 + \cdots + a_{jn}\epsilon_n$$

and therefore the i, j entry of the matrix of T^* is the j, i entry of A. In other words:

the matrix of T^* *is the transpose of the matrix of* T

where the matrix of T is taken with respect to the standard bases and the matrix of T^* is taken with respect ot he standard dual bases.

Now let's prove a few basic properties of duality.

Proposition 4. *Suppose that* $S : U \to V$ *and* $T : V \to W$*, so that* $TS : U \to W$*. Then* $(TS)^* = S^*T^*$*. (Both are maps* $W^* \to U^*$*).*

Proof. If $f \in W^*$, then

$$(TS)^*(f) = f \circ TS = (f \circ T) \circ S = S^*(T^*(f))$$

as desired. □

Proposition 5. *Suppose that* V, W *are finite dimensional.*

1. *If* $T : V \to W$ *is surjective, then* $T^* : W^* \to V^*$ *is injective.*
2. *If* $T : V \to W$ *is injective, then* $T^* : W^* \to V^*$ *is surjective.*

Proof. (1) Suppose that $f \in W^*$ and $T^*(f) = 0$, i.e., $f \circ T = 0$. Since T is surjective, if $w \in W$ then there exists $v \in V$ such that $Tv = w$. Then

$$0 = (f \circ T)(v) = f(T(v)) = f(w).$$

This proves that $f(w) = 0$ for all $w \in W$, so that $f = 0$. It follows that T^* is injective.

(2) Suppose $g \in V^*$. We must produce $f \in W^*$ such that $T^*(f) = g$. Let $v_1, \ldots, v_n$ be a basis of V; then the list $Tv_1, \ldots, Tv_n$ is linearly independent in W, so we can extend it to a basis $Tv_1, \ldots, Tv_n, w_1, \ldots, w_k$ of W. Now we can define $f : W \to \mathbf{F}$ by setting $f(Tv_i) = g(v_i)$ and $f(w_i) = 0$. Then $T^*(f) : V \to \mathbf{F}$ and

$$T^*(f)(v_i) = (f \circ T)(v_i) = f(Tv_i) = g(v_i),$$

so $T^*(f) = g$ as desired. □

Now the two previous propositions combine to give us the following.

Proposition 6. *Suppose that* V, W *are finite dimensional and* $T : V \to W$*. Then* $\dim \operatorname{image}(T) = \dim \operatorname{image}(T^*)$*.*

Proof. The map $T : V \to W$ can be written as a composition

$$V \xrightarrow{p} \operatorname{image}(T) \xrightarrow{\iota} W$$

where p is just the same as T (except its target space is image(T) instead of W) and ι is the inclusion of image(T) into W. Notice that p is surjective and ι is injective. Now by Proposition 4 the map $T^* : W^* \to V^*$ is the composition

$$W^* \xrightarrow{\iota^*} \operatorname{image}(T)^* \xrightarrow{p^*} V^*.$$

By Proposition 5, the map ι^* is surjective and the map p^* is injective. Since ι^* is surjective it follows that $\operatorname{image}(T^*) = \operatorname{image}(p^*)$, and so

$$\dim \operatorname{image}(T^*) = \dim \operatorname{image}(p^*) = \dim \operatorname{image}(T)^* = \dim \operatorname{image}(T)$$

where the second equality uses the fact that p^* is injective, and the third equality is the fact that taking the dual of a finite dimensional space doesn't change the dimension. □

Finally, as a special case of this proposition, we obtain our main result.

Corollary 7. *Let* A *be a matrix with entries in* $\mathbf{F}$*. Then the dimension of the column space of* A *is equal to the dimension of the row space of* A*.*

Proof. Let $T : \mathbf{F}^n \to \mathbf{F}^m$ be the linear transformation which has matrix A with respect to the standard basis. Then image(T) is the column space of A, and $\dim \operatorname{image}(T)$ is the dimension of the column space of A.

On the other hand, by Example 3, the dual map $T^* : (\mathbf{F}^m)^* \to (\mathbf{F}^n)^*$ has matrix A^t with respect to the standard dual bases (the superscript t denotes the transpose). It follows that $\dim \operatorname{image}(T^*)$ is the dimension of the column space of A^t, and this is certainly the dimension of the row space of A.

The result now follows by Proposition 6. $\square$