MATH 111: HW 1 SOLUTIONS AND COMMENTS

2.1 #1. If there are approximately 18 mangos per layer, and 5 layers, then there are approximately $18 \times 5 = 90$ mangos in the box.

2.1 #2. Take note that the question asks you **how** you would estimate the number of balls in the box. It does does not ask you to perform the estimate yourself, and it does not ask for a numerical answer.

In the hypothetical situation provided in the problem, it is not necessary to estimate the size of an individual ping-pong ball. In the problem, you are given enough ping-pong balls to fill one layer of the box, so you should do just that: fill one layer of the box, and count how many balls fit in that layer.

Then, estimate how many layers will fit in the box. For example, you can do this by seeing how many balls will stack vertically in the box. But since the box is a cube, you can do that by seeing how many balls will fit along an edge of the bottom.

Finally, multiply the number of layers by the number of balls that fit in the bottom layer, to estimate the number of balls that will fit in the box.

Estimating the size of an individual ping-pong ball will be much less accurate!

2.1 #3. Certainly one can fit five CDs onto the five shelves, one per shelf. However, if there are six CDs, then by the pigeonhole principle, if they are placed on five shelves then one shelf will have at least two CDs.

Please don't avoid the question by saying that CDs can be placed on the the top of the rack or underneath the rack. If the top or underneath counts as a shelf, then it should already be included in the 5 shelves. If they don't count as a shelf, then since the question asks about whether the CDs can be placed *on the shelves*, for the purposes of answering the question you can't use the top or the underneath!

2.1 #4. If five shelves each contain three CDs, then the total number of CDs is $3 \times 5 = 15$. If five shelves each contain three or fewer CDs, then there are at most 15 CDs on the shelves. Therefore, if 16 CDs are placed on the shelves, at least one shelf must have more than three CDs.

2.1 #6. It is not enough in this problem just to say "I think the bills would be too heavy": you are asked to justify your thinking quantitatively.

Here is one way to estimate the weight of the bills. We know from our experience that 400 sheets of papers weighs something in the range of 1 kilogram. Since a piece of paper is the size of roughly 3 one-dollar bills, we estimate that each bill weighs approximately 1 gram.

So one million bills weighs approximately $1,000,000$ grams, which equals $1,000$ kilograms. Certainly this is way too much to hold on your stomach!

Why is it not OK to say "I think the bills would be too heavy"? Well, if the question was about $100,000$ bills, then it looks like you might be able to do it. So you would be missing a big opportunity if you guessed no without thinking it through carefully! Should you really give up a million dollars by guessing that you couldn't do it, without trying to justify that guess?

Also note that it is not a good idea to *guess* the weight of a single dollar bill: if your guess is too low, then you might decide you could withstand the million dollars. If your guess is too high, then the same thought process might lead you mistakenly to pass up a smaller dollar amount. So your estimate of the weight of a dollar bill should be justified somehow.

2.1 #8. The first square contains $1 = 2^0 = 2^{1-1}$ pieces of gold. The second square contains $2 = 2^1 = 2^{2-1}$ pieces of gold. The third square contains $4 = 2^2 = 2^{3-1}$ pieces of gold. In general the nth square would contain 2^{n-1} pieces of gold. For example, the 64th square would contain

$$2^{63} = 9,223,372,036,854,775,808 \approx 9.223 \times 10^{18}$$

pieces of gold. Obviously, this is more gold than the king can provide!

All of the squares put together contain nearly twice as many pieces of gold as the final square, so approximately 1.8×10^{19} pieces of gold. In fact the exact answer is

$$18,446,744,073,709,551,616 = 2^{64} - 1$$

pieces of gold. (Can you see why it should be $2^{64} - 1$? How many pieces of gold are in the first two squares combined? In the first three squares combined? In the first n squares combined?)

Be careful when using a calculator on this problem: the numbers are too big for many calculators to handle. If you are going to use a calculator on a problem involving numbers so big that calculators cannot display completely, you **must** be careful to understand how your calculator handles big numbers. If as a result of misunderstanding your calculator you wrote down a number that was much too small, then you got this problem wrong.

2.1 #9. Here are some interesting answers.

- 29 is prime and 27 is not.
- 29 can be the number of days in a month, and 27 cannot.
- $29 = 2^2 + 5^2$ is a sum of two squares, and 27 is not.
- If you enter 29 on a calculator and turn it upside-down, you get a number again (62).

2.1 #10. One way to do this is by trial and error: for each number starting at 7, check to see if it is perfect: write down all of the number's divisors other than itself, add them up, and see whether it equals the number again. (For example, the divisors of 24 other than 24 are $1, 2, 3, 4, 6, 8, 12$, and these add up to 36, so 24 is not perfect.) By this method, the next perfect number that one finds is $28 = 1 + 2 + 4 + 7 + 14$.

Some of you noticed that since the problem says that no odd perfect numbers are known, you can skip odd numbers in the trial and error calculation. That's a reasonable thing to do.

Others found (maybe by searching on the web?) that there is a formula for the even perfect numbers: if $2^n - 1$ is prime, then $2^{n-1} \times (2^n - 1)$ is an even perfect number, and you get every even perfect number this way. For example, $n = 2$ gives $2^1 \times (2^2 - 1) = 2 \times 3 = 6$ is perfect, since 3 is prime. The next perfect number is $2^2 \times (2^3 - 1) = 4 \times 7 = 28$, since 7 is prime. (You can find more perfect numbers after that. $2^4 - 1 = 15$ is not prime, but $2^5 - 1 = 31$ is prime, so $2^4 \times 31 = 16 \times 31 = 496$ is the next perfect number after 28.

2.1 #11. Much much bigger than all of the options you are given! If you fold it 50 times, the thickness will be 2^{50} sheets, which is approximately 10^{15} sheets. (It is exactly

$$1,125,899,906,842,624$$

sheets.) If 100 sheets are approximately 1 centimeter, then $10,000$ sheets are 1 meter, so 10^{15} sheets are approximately 10^{11} meters. This equals 10^8 kilometers, or $100,000,000$ kilometers. That's most of the distance from the earth to the sun!

2.1 #15 Mathematicians suspect that you will always win, but no one knows how to prove it. This is known as the "Collatz Conjecture", or the "$3x + 1$ problem". It is known to be true for all numbers up to around 2×10^{16} (checked by computer).