# MATH 111: HW 11 SOLUTIONS AND COMMENTS

**8.1, #2**. The chance is 1/2 that the tooth fairy will leave \$1, and 1/2 that the tooth fairy will leave \$0.50. Therefore, the expected value of the tooth fairy's payoff is

$$\frac{1}{2} \times \$1.00 + \frac{1}{2} \times \$0.50 = \$0.75\,.$$

**8.1, #3**. Now the chance is 1/3 that the tooth fairy will leave \$1, 1/3 that the tooth fairy will leave \$0.80, and 1/3 that the tooth fairy will leave \$0.50. Therefore, the expected value of the tooth fairy's payoff is

$$\frac{1}{3} \times \$1.00 + \frac{1}{3} \times \$0.80 + \frac{1}{3} \times \$0.50 = \frac{1}{3} \times \$(1.00 + 0.80 + 0.50) = \frac{\$2.30}{3}\,,$$

which is \$0.7666..., or approximately 76.6 cents.

**8.1, #12**. The probability of rolling each number from 1 to 6 is equal to 1/6, so the expected value of your roll is:

$$\begin{aligned} & \left(\tfrac{1}{6} \times 1\right) + \left(\tfrac{1}{6} \times 2\right) + \left(\tfrac{1}{6} \times 3\right) + \left(\tfrac{1}{6} \times 4\right) + \left(\tfrac{1}{6} \times 5\right) + \left(\tfrac{1}{6} \times 6\right) \\ = & \ \tfrac{1}{6} \times (1 + 2 + 3 + 4 + 5 + 6) \\ = & \ 21/6 \\ = & \ 3.5\,. \end{aligned}$$

**8.1, #13**. **Solution 1: The standard solution.** The probability of rolling a combined total of 2 on two dice is 1/36; the probability of rolling a combined total of 3 on two dice is 2/36; the probability of rolling a combined total of 4 on two dice is 3/36; and so forth. The expected value of your roll of two dice is therefore:

$$\begin{aligned} & \tfrac{1}{36} \cdot 2 + \tfrac{2}{36} \cdot 3 + \tfrac{3}{36} \cdot 4 + \tfrac{4}{36} \cdot 5 + \tfrac{5}{36} \cdot 6 + \tfrac{6}{36} \cdot 7 + \tfrac{5}{36} \cdot 8 + \tfrac{4}{36} \cdot 9 + \tfrac{3}{36} \cdot 10 + \tfrac{2}{36} \cdot 11 + \tfrac{1}{36} \cdot 12 \\ = & \ \tfrac{2}{36} + \tfrac{6}{36} + \tfrac{12}{36} + \tfrac{20}{36} + \tfrac{30}{36} + \tfrac{42}{36} + \tfrac{40}{36} + \tfrac{36}{36} + \tfrac{30}{36} + \tfrac{22}{36} + \tfrac{12}{36} \\ = & \ 252/36 \\ = & \ 7\,. \end{aligned}$$

**Solution 2: The short, sneaky solution.** Consider your two dice separately. By 8.1 #12, the expected value of your roll on the first die is 3.5; similarly the expected value of your roll on the second die is 3.5. Therefore, the expected value of the total on the two dice must be $3.5 + 3.5 = 7$.

**8.1, #27**. Let us separate this game into two parts. First, you pay \$5 to play — this contributes a total of $-\$5$ to your expected value for the game (since you pay it no matter what). Next, you flip three coins and you receive money: if you see no heads (TTT, a 1/8 chance) you receive \$20; if you see one head (TTH, THT, HTT, a 3/8 chance) you receive \$5; if you see more than one head (HHT, HTH, THH, HHH, a 4/8 chance) you receive nothing; so the amount you expect to receive is

$$\frac{1}{8} \times \$20 + \frac{3}{8} \times \$5 + \frac{4}{8} \times \$0 = \$35/8 = \$4.375\,.$$

Therefore, in total, the expected value for the game is

$$-\$5.00 + \$4.375 = -\$0.625$$

or $-62.5$ cents.

Here's another way to do the computation. In the case where you see no heads, you gain \$15 net; when you see one head, you break even; when you see two or more heads, you lose \$5 net. Therefore, in total, the expected value is

$$\frac{1}{8} \times \$15 + \frac{3}{8} \times \$0 + \frac{4}{8} \times (-\$5) = -\$0.625\,.$$

Notice that there is no single outcome in which you lose \$0.625. The expected value is not necessarily equal to one of the possible outcomes. Rather, an expected value is an average: the average value that you expect to gain (or lose) each time you play the game, if you were to play it many times.

**8.2, #3**.

$$280,000,000 \times 2\% = 280,000,000 \times 0.02 = 5,600,000 \text{ people}\,.$$

**8.2, #5**. A 12-year-old who dies instead of living to the age of 78 loses 66 years of life expectancy; this happens 7% of the time. Therefore, the expected loss of life expectancy is

$$7\% \times 66 = 0.07 \times 66 = 4.62 \text{ years}\,.$$

**8.2, #6**. Using the numbers given in the problem, we can calculate the following quantities:

- The fraction of blondes who are natural blondes, and who we believe correctly to be natural blondes: $0.9 \times 0.8 = 0.72$.
- The fraction of blondes who are natural blondes, and who we believe incorrectly to be bleached blondes: $0.9 \times 0.2 = 0.18$.
- The fraction of blondes who are bleached blondes, and who we believe correctly to be bleached blondes: $0.1 \times 0.8 = 0.08$.
- The fraction of blondes who are bleached blondes, and who we believe incorrectly to be natural blondes: $0.1 \times 0.2 = 0.02$.

Therefore the total fraction of blondes who we believe to be bleached blondes (correctly or incorrectly) is $0.18 + 0.08 = 0.26$.

We wish to compute the chances we are wrong in our belief that Chris is a bleached blonde. This probability is equal to:

$$\frac{\text{the fraction of blondes who we believe } \textbf{incorrectly} \text{ are bleached blondes}}{\text{the fraction of blondes who we believe are bleached blondes}}$$

which, according to the numbers we have calculated, is

$$0.18/0.26 \approx 69.2\% .$$

Therefore the chance that we are wrong in our belief about Chris is approximately 69%.

Notice that in this problem we only really needed to compute the second and third items in the list of quantities above — i.e. the cases where we believed someone to be a bleached blonde. The other two numbers never came into the computation.

**8.2, #10**. The principle in this problem is exactly the same as the principle in the previous problem. We use the shortcut that we noticed at the end of the last solution: since the professor recalls that you are a winner, we only care about the cases where the professor recalls you are a winner, and so we compute:

- The number of applicants who won the scholarship who the professor correctly recalls won the scholarship: $200 \times 90\% = 180$.
- The number of applicants who did not win the scholarship but who the professor incorrectly recalls won the scholarship: $(100,000 - 200) \times 10\% = 99,800 \times 10\% = 9,980$.

The total number of people who the professor recalls won the scholarship is $180 + 9,980 = 10,160$. Therefore, the probability that we actually won the scholarship is

$$\frac{\text{the number of people who the professor } \textbf{correctly} \text{ remembers won the scholarship}}{\text{the number of people who the professor remembers won the scholarship}}$$

which is $180/10160 \approx 1.77\%$. Surprisingly small!

**8.2, #15**. Let $c$ denote the total number of cell-phone users. (It is possible to avoid using a variable, if you like, by picking any number you like — other than zero! — to be the total number of cell-phone users. The variable will cancel out in the end.) Since 1 out of every 100,000 cell-phone users will die due to cell-phone use, the total number of people who will die due to cell-phone use is

$$c \times \frac{1}{100,000} = \frac{c}{100,000} .$$

A new safety feature may or may not save all of these lives, but in any case, the largest number of lives that a new safety feature could possibly save is the number of cell-phone related deaths — so the largest possible number of lives saved is $c/100,000$. For each life saved, the government is willing to spend \$12,500; therefore, the most the government is willing to spend is

$$\frac{c}{100,000} \times \$12,500 = \$(c/8)$$

since $100,000 = 12,500 \times 8$.

Finally, according to the statement of the problem, the number that we want to calculate is the amount that the government is willing to spend *per cell phone*, which is

$$\frac{\text{amount the government is willing to spend}}{\text{number of cell phones}} = \frac{c/8}{c} = \frac{1}{8} = \$0.125\,,$$

or 12.5 cents.