# MATH 111: HW 3 SOLUTIONS AND COMMENTS

**3.2 #2**. I graded this problem relatively strictly: some descriptions are better than others, and I only accepted descriptions which are *precise and complete*: that is, descriptions which fully describe the set in question, and which can describe *only* the set in question. For example, for the set $\{1, 2, 3, 4, 5\}$ it is true but unacceptable to say "This set consists of natural numbers": there are many other sets that fit this description.

(1) This is the set of multiples of 3 excluding the number 9. (I think the fact that 9 is missing is a typo; I didn't penalize anyone for not noticing that 9 is missing, so "This is the set of multiples of 3" was acceptable.)
(2) This set consists of the first five natural numbers.
(3) This set consists of the reciprocals of the powers of 2 (starting from 1/2, not 1/1).
(4) This set consists of the negative integers.
(5) This set consists of the first ten positive square numbers. (Alternately: the squares of the numbers 1 through 10.)

**3.2 #4**. Finite; infinite; finite; infinite.

**3.2 #8**. As in the first problem, some descriptions are better than others. Making a table which shows the beginning of the correspondence is a start, but is not a complete answer. (I gave this half-marks, which I felt was generous.) To give a complete description of a one-to-one correspondence between the natural numbers and EIF, you should explain how to figure out the member of EIF to which a given natural number corresponds.

The number 1 corresponds to 5, which is 5 less than 10; the number 2 corresponds to 15, which is 5 less than 20; the number 3 corresponds to 25, which is 5 less than 30; in general, the number $n$ will correspond to $10n - 5$ (equivalently, to $10(n - 1) + 5$).

Here is another way to say the same thing: given the number $n$, subtract 1 from it, and add a 5 at the end. For example, to see what 7 corresponds to, subtract 1 from 7 — that gives you 6—then put a 5 at the end—so, 65.

**3.2 #12**. The correspondence starts as follows: $1 \leftrightarrow 1$, $2 \leftrightarrow 3$, $3 \leftrightarrow 5$, $4 \leftrightarrow 7$, $5 \leftrightarrow 8$, $6 \leftrightarrow 10$, $7 \leftrightarrow 11$, and so on. Notice that 6 corresponds to $10 = 6 + 4$ and 7 corresponds to $11 = 7 + 4$. In fact, from this point onwards, all larger numbers are in OWS, so a natural number $n$ (greater than or equal to 6) should correspond to the number $n + 4$ in OWS.

**3.2 #21**. If a set is finite, then removing some of its elements will certainly give you a smaller set. However, *if a set is infinite*, then removing elements does not necessarily leave you with a set of smaller cardinality. (For example, the set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$ and the set $\{2, 3, 4, \ldots\}$ — the natural numbers with 1 removed — are in one-to-one correspondence by adding 1 to each natural number.)

**3.3 #3**. The number is constructed by concatenating (stringing together) all the natural numbers, in order. The 14th digit (after the decimal point) is 1, the 25th digit is 7, and the 31st digit is 0).

**3.4 #4**. Any number whose first digit is not 3 cannot be the first number on her list; any number whose second digit is not 8 cannot be the second number on her list; and any number whose third digit is not 2 cannot be the third number on her list. Therefore any number with all three of these properties cannot be any of the three numbers on her list.

For example, 11111 is not on the list: it can't be the first number (its first digit is 1 instead of 3); it can't be the second number (its second digit is 1 instead of 8); it can't be the third number (its third digit is 1 instead of 2). In fact, no number from 11100 to 11199 can be on the list. (Can you see why?)

**3.4 #13**. Let $B$ denote a circle which is coloured blue, and let $R$ denote a circle which is coloured red. We will use the diagonalization argument to show that the set of all possible circle colourings cannot be put into one-to-one correspondence with the natural numbers.

Suppose that we attempted to construct a one-to-one correspondence between the natural numbers and the set of circle colourings. Our correspondence might begin:

$$\begin{array}{ccl} 1 & \leftrightarrow & \underline{B}RBRBRR\cdots \\ 2 & \leftrightarrow & R\underline{R}BRBRR\cdots \\ 3 & \leftrightarrow & RR\underline{R}RBBB\cdots \\ \vdots & & \vdots \end{array}$$

Now we wish to show that our attempt to construct a one-to-one correspondence must have failed: that is, we must have missed at least one circle colouring. We use the diagonalization argument: in the example above, any colouring which begins with $R$ does not correspond to 1; any colouring whose second circle is coloured $B$ does not correspond to 2; any colouring whose third circle is coloured $B$ does not correspond to 3; and so forth.

That is: the colouring which begins $RBB\cdots$ and whose $n$th circle is coloured the opposite of the colour of the $n$th circle in the colouring which corresponds to $n$, is missing from the list.

Let me stress two crucial points. First: the diagonalization argument works because we've taken the diagonal and *changed* it. We're making a colouring which is different from all the ones in the correspondence, by making a colouring which is *guaranteed to disagree in at least one place with each colouring in the correspondence.*

Second: this doesn't work by just picking one colouring in the correspondence, and changing it in one place. For example, if you take the colouring $BRBRBRR\cdots$ which corresponds to 1, and you change the colour of the first circle from $B$ to $R$, then of course you no longer have the colouring which corresponds to 1; but nothing is stopping the new sequence from occuring somewhere else in the list! (Indeed, $RRBRBRR\cdots$ could correspond to 2.) You really have to go diagonally and make something which differs from *every single colouring in the correspondence.*

**3.3 #14**. This question is identical to the previous one!! (Just change the letters $B$ and $R$ to $H$ and $T$!)

**3.3 #18**. To answer this question satisfactorily, you must explain a procedure which will give three different real numbers that are not on the list. There are many different ways to do this; here are several. Let's suppose our list of real numbers begins like this:

$$\begin{array}{ccl} 1 & \leftrightarrow & 0.\underline{7}3498344\cdots \\ 2 & \leftrightarrow & 0.3\underline{3}102983\cdots \\ 3 & \leftrightarrow & 0.43\underline{5}53894\cdots \\ 4 & \leftrightarrow & 0.140\underline{9}2109\cdots \\ \vdots & & \vdots \end{array}$$

Here's the first way. The book uses the following procedure to construct one real number not on the list: if the diagonal digit is not a 2, change it to a 2; if the diagonal digit is a 4, change it to a 4. In the above example, this produces a real number beginning $0.2222\cdots$. One can make different numbers by using different digits instead of 2 and 4: for example, using 3 and 6, one gets $0.3633\cdots$. Using 5 and 7, one gets $0.5575\cdots$.

Three comments on this method. First, you have to be careful to use pairs of digits with no overlap at all. For example, if you decided to use 2 and 1 instead of 2 and 4, then the diagonalization method would still give you a number beginning $0.2222\cdots$, which might not be different from the first number you produced! Second (see problem #21 in this section), problems can arise if you use the digit 9, so the digit 9 must be avoided. Finally, this method has a serious drawback: because of the previous comments, this method can produce at most four different real numbers not on the list! Let's look at other methods which will let us produce as many different real numbers as we like which are not on the list.

Some people had a very nice idea: let's go back to our first method, with the 2s and 4s. After we've used the diagonalization method once (producing $0.2222\cdots$, like we saw before), let's look at the diagonal starting from the *second* digit of the first number.

$$\begin{array}{ccl} 1 & \leftrightarrow & 0.7\underline{3}498344\cdots \\ 2 & \leftrightarrow & 0.33\underline{1}02983\cdots \\ 3 & \leftrightarrow & 0.435\underline{5}3894\cdots \\ 4 & \leftrightarrow & 0.1409\underline{2}109\cdots \\ \vdots & & \vdots \end{array}$$

Then any number which has 2 as the second digit after the decimal point is different from $0.73498344\cdots$; any number which has 2 as its third digit is different from $0.33102983\cdots$; any number which has 2 as its fourth digit is different from $0.43553894\cdots$; and so on. In this manner, we construct a number beginning $0.?2224\cdots$ which is not on our original list. (The ? means we haven't picked a first digit yet.)

But be careful! We're not done yet! The diagonal argument guarantees that the number we just made isn't on our original list, but nothing is guaranteeing that it isn't equal to the first different number that we made! After all, the first number we made was $0.2222\cdots$, and the new number is $0.?2224\cdots$, which might not be different if we pick the ? badly. So we pick the ? intelligently: the first number we made ($0.2222\cdots$) started with a 2 , so if we replace the ? with a 4 then the new number is guaranteed to be different.

Now do this again using the third diagonal, and using the first two digits to guarantee that the resulting number is different from the first two numbers we made.

Well, that sounded really complicated — but actually, there's a very very easy way to explain it. The idea is just this: start with out initial list of numbers, and use the diagonal argument to produce a number which isn't on it. Then *add that new number at the top of the list, and use the diagonal argument again on the new list.* In the example, after we've made $0.2222\cdots$, add this to the top of the list:

$$\begin{array}{ccl} 1 & \leftrightarrow & 0.\underline{2}222\cdots \\ 2 & \leftrightarrow & 0.7\underline{3}498344\cdots \\ 3 & \leftrightarrow & 0.33\underline{1}02983\cdots \\ 4 & \leftrightarrow & 0.435\underline{5}3894\cdots \\ 5 & \leftrightarrow & 0.1409\underline{2}109\cdots \\ \vdots & & \vdots \end{array}$$

and use the diagonal argument to get $0.42224\cdots$. Then add that to the top of the list:

$$\begin{array}{ccl} 1 & \leftrightarrow & 0.\underline{4}2224\cdots \\ 2 & \leftrightarrow & 0.2\underline{2}22\cdots \\ 3 & \leftrightarrow & 0.73\underline{4}98344\cdots \\ 4 & \leftrightarrow & 0.331\underline{0}2983\cdots \\ 5 & \leftrightarrow & 0.4355\underline{3}894\cdots \\ 6 & \leftrightarrow & 0.14092\underline{1}109\cdots \\ \vdots & & \vdots \end{array}$$

and use the diagonal argument to get $0.242222\cdots$. Then add that to the top... obviously, we can keep going like this forever!