Induction

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Induction. Let $P(x)$ be a statement about x. In order to show that $P(n)$ is (eventually) true for all possible integers n greater than some starting point, a, one can proceed as follows:

- (The "first" case) Show there is some integer a such that $P(a)$ is true.
- (The Inductive Hypothesis) Assume the statement is true for integers up to $k \geq a$ (on down to the first case $P(a)$).
- (The Inductive Step) Prove that the "next" case, $P(k+1)$, is true using the first case and the Inductive Hypothesis.

Example 1. Prove for all $n \geq 1$

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Example 2. If V, E, and F represent (respectively) the number of vertices, edges, and faces of a connected planar graph, then

$$V - E + F = 2.$$

Example 3. Use induction to show that for all $n \geq 0$ that

$$\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30} \in \mathbb{Z}.$$

Example 4 (Recurrence). A (fair) coin is tossed n times. What is the probability that two heads appear in succession somewhere in the sequence of throws?

Example 5 (General Induction). Let F_k denote the kth Fibonacci number, prove

$$F_{n+1}^2 + F_n^2 = F_{2n+1}.$$

Problem 1. If $\{a_n\}$ is a sequence such that for $n \geq 1$, $(2 - a_n)a_{n+1} = 1$, what happens to a_n as n tends towards infinity?

Problem 2. Use induction to prove that there are exactly 2^n subsets of a set containing n elements.

Problem 3. Show that every number in the following sequence is divisible by 53:

$$1007, 10017, 100117, 1001117, \ldots \ .$$

Problem 4. Suppose n coins are given, named C_1, C_2, ..., C_n. For each k, C_k is biased so that, when tossed, it has probability $\frac{1}{2k+1}$ of falling heads. If the n coins are tossed, what is the probability that the number of heads is odd? Express the answer as a rational function of n. (Putnam 2001).

Problem 5. Let r be a number such that $r + \frac{1}{r}$ is an integer. Prove that $r^n + \frac{1}{r^n}$ is an integer for every positive integer n.

Problem 6. Let a_1, a_2, ..., a_n be a permutation of the set $S_n = \{1, 2, \ldots, n\}$. An element i in S_n is called a fixed point of this permutation if $a_i = i$.

1. A *derangement* of S_n is a permutation of S_n having no fixed points. Let g_n denote the number of derangements of S_n. Show that

$$g_1 = 0,\ g_2 = 1,$$

and

$$g_n = (n-1)(g_{n-2} + g_{n-1}) \text{ for } n > 2.$$

2. Let f_n be the number of permutations of S_n with exactly 1 fixed point. Show that

$$|f_n - g_n| = 1.$$

Problem 7. Let P_n denote the number of regions formed when n lines are drawn in the Euclidean plane in such a way that no three lines meet at one point and no two lines are parallel. Come up with a recurrence relation for P_n, and prove that it holds for all $n \geq 1$.

Problem 8. Prove the arithmetic-mean-geometric-mean inequality, which states, for a_1, ..., a_n all positive real numbers, that

$$\frac{a_1 + \ldots + a_n}{n} \geq (a_1 \cdot \ldots \cdot a_n)^{\frac{1}{n}}.$$

Problem 9. Let n be a positive integer, and $a_i \geq 1$, for $i = 1, 2, \ldots, n$. Show that

$$(1 + a_1)(1 + a_2) \cdots (1 + a_n) \geq \frac{2^n}{n+1}(1 + a_1 + \ldots + a_n).$$

Problem 10. Let F_k be the kth Fibonacci number, prove for all positive integers that

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

Problem 11. Suppose $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are real numbers, prove

$$\sum_{k=1}^{n} \sqrt{a_k^2 + b_k^2} \geq \sqrt{\left(\sum_{k=1}^{n} a_k \right)^2 + \left(\sum_{k=1}^{n} b_k \right)^2}.$$

Problem 12. Show that for $n \geq 6$ a square can be dissected into n smaller squares, not necessarily all of the same size.