MATH 294A (PUTNAM SEMINAR): PARITY AND INVARIANTS

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Many problems involve configurations which are changing in complicated ways. An important technique is to look for an *invariant*, an auxiliary quantity which either does not change when the configuration changes, or changes in a way that is easy to understand (e.g. strictly increasing or strictly decreasing). The behavior of this auxiliary quantity can give you information about the achievable configurations.

The most common and important example is *parity*: if some quantity is either *always even* or *always odd*, then the parity of the initial configuration tells you the parity of the final configuration.

Example 1 (A classic). The opposite corners of a chessboard are removed. Is it possible to cover the remaining 62 squares using 2-by-1 dominos?

Example 2. In the mathematics department, every faculty member has at most three enemies. (Assume that being an enemy is symmetric: if Professor X is an enemy of Professor Y, then Professor Y is an enemy of Professor X.) Prove that the faculty can be divided into two committees such that each faculty member has at most one enemy on their committee.

Example 3. Is it possible for two different powers of 2 to have the same digits (in a different order)?

Example 4. Suppose that not all four integers a, b, c, d are equal. Start with (a, b, c, d) and repeatedly replace (a, b, c, d) by $(a-b, b-c, c-d, d-a)$. Show that eventually at least one number of the quadruple will become arbitrarily large.

Problem 1. Start with the positive integers $1, \ldots, 4n-1$. In one move you may replace any two integers by their difference. Prove that after $4n-2$ steps, the final remaining integer will be even.

Problem 2. 64 coins are arranged in an 8-by-8 square. Initially half the coins are heads-up and half the coins are tails-up, in an alternating pattern. At each step there are three possible moves: flip all eight coins in a single row; flip all eight coins in a single column; flip all four coins in a 2-by-2 square. Is it possible to reach a configuration where exactly one coin is heads-up and the rest are tails-up?

Problem 3. A circle is divided into six sectors. The numbers $1, 0, 1, 0, 0, 0$ are written into the sectors (in counterclockwise order). At each step you may increase two neighboring numbers by 1. After a sequence of such steps is it possible for all the numbers to be equal?

Problem 4. Three bugs are crawling on the coordinate plane. They move one at a time, and each bug will only crawl in a direction parallel to the line between the other two. If the bugs start out at $(0,0),(3,0),(0,3)$, can the bugs end up at $(1,2),(2,5),(-2,3)$? Is it possible that after some time the first bug will end up back where it started, while the other two bugs have switched places?

Problem 5. Start with the set $\{3,4,12\}$ at each step you may choose two of the numbers a,b and replace them by $0.6a-0.8b$ and $0.8a+0.6b$. Can you reach the set $\{4,6,12\}$? Can you reach any set $\{x,y,z\}$ with $|x-4|,|y-6|,|z-12|$ all less than $1/\sqrt{3}$?

Problem 6. In a certain small town there live n fickle dwarves, each of whom lives in a red house or a blue house. Every day, one dwarf looks at all the houses other than his own; if he discovers that the color of his own house is less popular among the $n-1$ other houses than the other color, he repaints his house in the other color. Prove that eventually there will be no more repainting.

Problem 7. The vertices of an n-gon are labeled by real numbers $x_1,\ldots,x_n$. Let a,b,c,d be four successive labels. If $(a-d)(b-c)<0$, we may swap b with c. Can this switching operation be performed infinitely many times?

Problem 8. Start with the integer 7^{2006}. At each step, delete the leading digit, and add it to the remaining number. This is repeated until a number with exactly 10 digits remains. Prove that this number has two equal digits.

Problem 9. There is a checker at the point $(1,1)$ in the plane. At each step, you may move the checker in one of two ways: by doubling one of the two coordinates, or (if the coordinates are unequal) by subtracting the smaller coordinate from the larger one. Is it possible for the checker to end up at the point $(3,3)$?

Problem 10. n numbers are written on a blackboard. At each step, two numbers a,b are erased, and $\frac{a+b}{4}$ is written on the board. If the initial n numbers are all equal to 1, prove that after $n-1$ steps, the one remaining number is at least $\frac{1}{n}$.

Problem 11. There is a row of 1000 integers. A second row of integers is constructed underneath it, as follows: underneath the integer a, write the number of times that a occurs in the first row. In the same way, we produce a third row by counting occurrences in the second row, and so forth. Prove that eventually one of the rows is identical to the subsequent row.

Problem 12. Nine squares in a 10-by-10 grid are infected. In one time unit, the squares which share an edge with at least two infected squares also become infected. Can the infection spread to the whole square?