Elementary Number Theory

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One of number theory's claims to fame is the unusual ease with which one can pose exceedingly difficult problems, and the enormity of the full toolkit needed to tackle all of these problems. Fortunately, the good news is that solving a Putnam problem is rarely about having memorized the applicable theorem. Instead, there are a select few elementary results/topics that can cover a wide range of possible questions:

- Modular arithmetic.
- Euler's theorem (in particular Fermat's Little Theorem).
- Prime factorization, gcd's, and divisibility.

Example 1. Show that the sequence

$$11, 111, 1111, 11111, \cdots$$

contains no perfect squares.

Example 2. Prove that the fraction

$$\frac{21n+4}{14n+3}$$

is irreducible for every positive integer n.

Example 3. What are the last two digits of $3^{3^{2006}}$?

Example 4. Suppose that the number of prime divisors of a positive integer n is a prime number p which does not divide n. Show that n is one more than a multiple of p.

Problem 1. If $2n+1$ and $3n+1$ are both perfect squares, show that n is divisible by 40.

Problem 2. How many trailing zeros are at the end of the decimal expansion of $\binom{150}{72}$?

Problem 3. Find all positive integers d such that d divides both n^2+1 and $(n+1)^2+1$ for some n.

Problem 4. For any prime p, prove that every prime divisor of 2^p-1 is at least p.

Problem 5. Prove that for any integers m and n, the quantity

$$\frac{gcd(m,n)}{n}\binom{n}{m}$$

is an integer.

Problem 6. Let p_k denote the k-th prime number. Show that

$$p_k < 2^{2^k}.$$

Problem 7. Prove that for any integer k, the number $n = 9^k \cdot 2006 + 1$ cannot be expressed in the form

$$n = x^2 + y^2 + z^2$$

for any integers x, y, and z.

Problem 8. Count the number of pairs of positive integers (x, y) such that

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{2006}.$$

Problem 9. Find the sum of the digits of the sum of the digits of the sum of the digits of 2006^{2005}.

Problem 10. Show that for any prime p, the number $2^p + 3^p$ is never a perfect power (greater than 1) of an integer.

Problem 11. Show there are no non-trivial (i.e. other than $(x, y, z) = (0, 0, 0)$) integer solutions to the equation

$$x^3 + 3y^3 + 9z^3 - 9xyz = 0.$$

Problem 12. For a given positive integer m, find all triples (n, x, y) of positive integers with m and n relatively prime, which satisfy the relation

$$(x^2 + y^2)^m = (xy)^n.$$