The Pigeonhole Principle*

Nick Rogers

August 28, 2006

Pigeonhole Principle. If $n+1$ pigeons are distributed among n holes, then at least one of the holes has two or more pigeons.

Example 1. Show that if five points are placed in a 1×1 square, then there is a pair of points separated by a distance of no more than $\frac{\sqrt{2}}{2}$.

Example 2. Suppose that $n+1$ distinct numbers are chosen in the range $1, 2, \ldots, 2n$.

(a) Show that there is a pair of numbers that are relatively prime (i.e., have no factors in common).

(b) Show that there is a pair of numbers such that one divides the other.

Example 3. Show that any set of 10 distinct integers in the range $1, 2, \ldots, 99$ has two disjoint nonempty subsets with the same sum.

Example 4. Suppose $\alpha \in \mathbb{R}$ is irrational. Show that there are infinitely many rational numbers $\frac{p}{q}$ such that

$$\left| \frac{p}{q} - \alpha \right| \leq \frac{1}{q^2}.$$

Generalized Pigeonhole Principle. If $kn+1$ pigeons are distributed among n holes, then at least one of the holes has $k+1$ or more pigeons.

Example 5. Suppose that 55 numbers are chosen in the range $1, 2, \ldots, 100$. Show that there is a pair of numbers that differ by 9.

*I am indebted to *Problem-Solving through Problems* by Loren C. Larson, from which many of these problems are drawn.

Problem 1. Suppose that 5 points are placed in an equilateral triangle whose side length is 1. Show that there is a pair of points separated by a distance of no more than $\frac{1}{2}$.

Problem 2. Show that if there are n people at a party, then two of them know the same number of people (among those present).

Problem 3. Suppose that 20 numbers are chosen from the arithmetic progression $1, 4, 7, \ldots, 100$. Show that there are two distinct integers in A whose sum is 104.

Problem 4. Show that in any group of six people there are either three mutual friends or three mutual strangers. (*Hint:* given any of the six people, he or she must either have three friends or three strangers in the group. Now consider the relationships among these four people.)

Problem 5. Suppose that 19 points are placed in a regular hexagon with side length 1. Show that there is a pair of points separated by a distance of no more than $\frac{\sqrt{3}}{3}$.

Problem 6. Suppose that 17 people correspond by email, each one with all the rest. Each pair discusses one of three possible topics: politics, science, or religion. Show that there are at least three people who all correspond with each other about the same topic. (*Hint:* this is like Problem 4, except with three possibilities for each pair of people instead of two.)

Problem 7. Suppose that $f(x)$ is a polynomial with integer coefficients, and $f(a) = f(b) = f(c) = 2$ for distinct integers a, b, and c. Show that there is no integer d so that $f(d) = 3$. (*Hint:* show that $f(p) - f(q)$ is divisible by $p - q$.)

Problem 8. Show that in any set of n integers, there is a nonempty subset whose sum is divisible by n.

Problem 9. Suppose that 55 numbers are chosen in the range $1, 2, \ldots, 100$. We have seen that there is a pair that differ by 9. Is there necessarily a pair that differs by 10? 11? 12? 13?

Problem 10. Suppose each point of the plane is colored red or blue. Show that there is a rectangle all of whose vertices are the same color.

Problem 11. Suppose that 64 dots are arranged in an 8×8 grid. Is it possible to divide the plane with 13 lines so that each region contains at most one of these points?

Problem 12. Given any sequence of $mn + 1$ real numbers, show that there is an increasing subsequence of length $m + 1$ or a decreasing subsequence of length $n + 1$.