PROBLEM SOLVING SEMINAR: GENERATING FUNCTIONS

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If $\{a_n\}_{n\geq 0}$ is a sequence, one can form the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n .$$

This power series is called the *generating function* of the sequence $\{a_n\}$, and one can discover properties of the sequence by studying properties of the power series $f(x)$. This is an extremely powerful technique, and we will barely skim the surface in these problems.

The simplest generating function is the generating function for the geometric sequence $\{c^n\}$:

$$\sum_{n=0}^{\infty} c^n z^n = \frac{1}{1-cz} ,$$

and this series converges for complex numbers $|z| < |1/c|$.

Example 1. Suppose c, r are real constants. Prove more generally that one has

$$\sum_{n=0}^{\infty} \binom{r+n}{n} c^n z^n = \frac{1}{(1-cz)^{r+1}} ,$$

where $\binom{r+n}{n}$ is defined to be

$$\frac{(r+n)(r+n-1)\cdots(r+1)}{n!} ,$$

and again this equality is valid when $|z| < |1/c|$.

Here is a very common application of this technique. If one has a sequence that is defined recursively, one may multiply both sides of the recursive formula by z^n, and sum over n. This gives an equation for the generating function, and knowledge of the generating function often helps one find a closed form for the sequence. Often the generating function will turn out to be a ratio of polynomials, in which case one uses:

Theorem: (Partial fractions) Suppose $f(z), g(z)$ are polynomials with $g(0) \neq 0$ and $\deg(f) < \deg(g)$, and write $g(z) = c(1-r_1 z)^{m_1} \cdots (1-r_d z)^{m_d}$. Then there exist polynomials $p_1, \ldots, p_d$ with $\deg(p_i) < m_i$ such that the coefficient of z^n in the power series expansion of $\frac{f(z)}{g(z)}$ has the form

$$p_1(n) \cdot r_1^n + \cdots + p_d(n) \cdot r_d^n .$$

Example 2. Find a closed form for the recurrent sequence g_n with $g_0 = g_1 = 1$ and $g_n = g_{n-1} + 2g_{n-2} + (-1)^n$ for $n \geq 2$.

The *convolution* of two sequences corresponds to the *product* of their generating functions: if $f(z) = \sum_{n\geq 0} a_n z^n$ and $g(z) = \sum_{n \geq 0} b_n z^n$ then $f \cdot g = \sum_{n\geq 0} c_n z^n$ where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} \,.$$

Example 3. Prove that

$$\binom{n}{j} = \sum_{k=0}^{2j} (-1)^{j+k} \binom{n}{k} \binom{n}{2j-k} \,.$$

In some situations it is advantageous to consider the *exponential generating function* $\sum_n \frac{a_n}{n!} z^n$. For exponential generating functions, the convolution becomes: if $f(z) = \sum_{n\geq 0} \frac{a_n}{n!} z^n$ and $g(z) = \sum_{n\geq 0} \frac{b_n}{n!} z^n$ then $f \cdot g = \sum_{n \geq 0} \frac{c_n}{n!} z^n$ where

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \,.$$

Problem 1. Prove that the generating function for the sequence $\{\binom{r}{n} c^n\}_{n\geq 0}$ is $(1 + cz)^r$. Here c, r are arbitrary real constants.

Problem 2. Evaluate $\sum_{n=0}^{\infty} H_n/10^n$, where $H_n = \sum_{k=1}^{n} \frac{1}{k}$.

Problem 3. In how many ways can a 2-by-2-by-n pillar be built out of $2 \times 1 \times 1$ bricks?

Problem 4. Solve the recurrence $g_0 = 0$, $g_1 = 1$,

$$g_n = -2ng_{n-1} + \sum_k \binom{n}{k} g_k g_{n-k} \,.$$

Problem 5. Suppose that $z/(e^z - 1)$ is the exponential generating function for the sequence B_n. Prove that $B_{2k+1} = 0$ for $k > 0$.

Problem 6. Let $[a, b]$ denote the arithmetic progression $\{b, a + b, 2a + b, \ldots\}$. Suppose that $[a_1, b_1], \ldots, [a_k, b_k]$ are pairwise disjoint and $\cup_{i=1}^{k} [a_i, b_i] = \mathbf{N}$. If $2 \leq a_1 \leq a_2 \leq \cdots \leq a_k$, prove that $a_{k-1} = a_k$.

Problem 7. A certain sequence g_n satisfies the recurrence

$$ag_n + bg_{n+1} + cg_{n+2} + d = 0$$

for $n \geq 0$ and integers a, b, c, d with $\gcd(a, b, c, d) = 1$. It also has the closed form $g_n = \lfloor \alpha(1 + \sqrt{2})^n \rfloor$ with $0 < \alpha < 1$. Find all possibilities for a, b, c, d, α.

Problem 8. Let C_n be given by the recurrence formula $C_0 = 1$ and $C_n = C_{n-1}C_0 + C_{n-2}C_1 + \cdots + C_0 C_{n-1}$ for $n \geq 1$. Prove that the generating function for C_n is

$$\frac{1-\sqrt{1-4z}}{2z}$$

and use this to compute a closed form for C_n.

Problem 9. Let $\left\{ {n \atop k} \right\}$ denote the number of unordered partitions of n into k nonempty subsets; for instance, $\left\{ {4 \atop 2} \right\} = 7$. If $n \geq 0$ prove that

$$\left\{ {n \atop k} \right\} = k \left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\} .$$

Problem 10. Prove that

$$\left\{ {n \atop k} \right\} = \sum_{r=1}^{k} (-1)^{k-r} \frac{r^n}{r!(k-r)!} .$$

Problem 11. The nth Bell number b_n is the total number of unordered partitions of n into (any number of) nonempty subsets. Prove the formula

$$b_n = \frac{1}{e} \sum_{r=0}^{\infty} \frac{r^n}{r!} .$$

Problem 12. Prove that the exponential generating function for b_n is $e^{e^z-1} - 1$.