Geometry*

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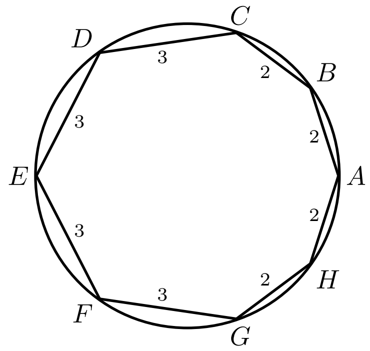
Example 1. (1978 Putnam Exam) Find the area of a convex octagon that is inscribed in a circle and has four consecutive sides of length 2 and the remaining four sides of length 3.



Example 2. Show that the diagonals of a quadrilateral $ABCD$ are orthogonal if and only if the sums of the squares of opposite sides are equal:

$$\overline{AC} \perp \overline{BD} \Leftrightarrow (AB)^2 + (CD)^2 = (AD)^2 + (BC)^2.$$

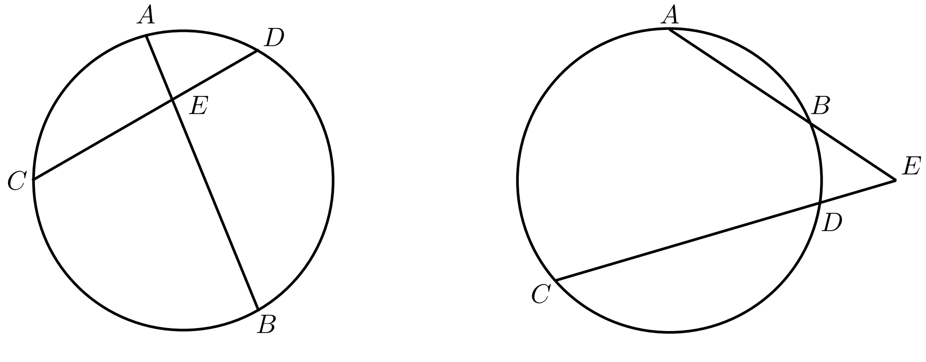
Example 3. Suppose that a regular n-gon is inscribed in a circle, and P is a point on the circumference of that circle. Label the vertices of the n-gon $A_1, A_2, \ldots, A_n$. Show that the sum

$$(A_1P)^2 + (A_2P)^2 + \cdots + (A_nP)^2$$

is independent of P.

Problem 1. Show that, in each of the following figures,

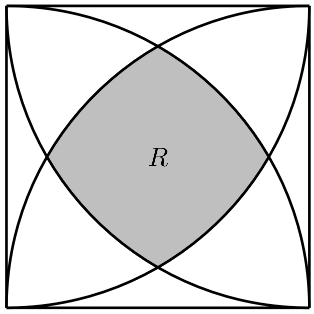
$$AE \cdot BE = CE \cdot DE.$$



Problem 2. Find the exact values of cos(36°) and sin(36°). *Hint:* You may find it helpful to consider a 36°-36°-108° triangle, or a regular pentagon.

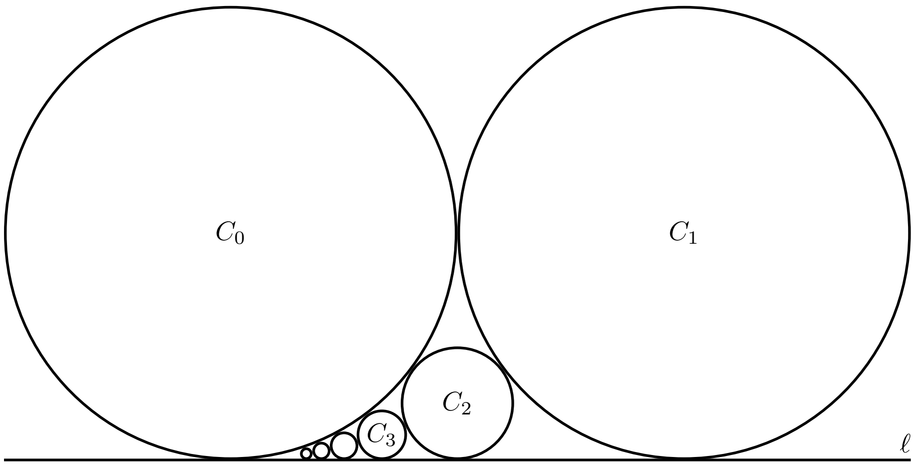
*Many of these problems are adapted from those in *Problem-Solving through Problems* by Loren C. Larson, and *Problem-Solving Strategies*, by Arthur Engel.

Problem 3. (Arlo Caine, Math 223 Course Materials) In the following diagram, an arc of radius 1 is centered at each of the corners of a unit square. Find the area of the region R.



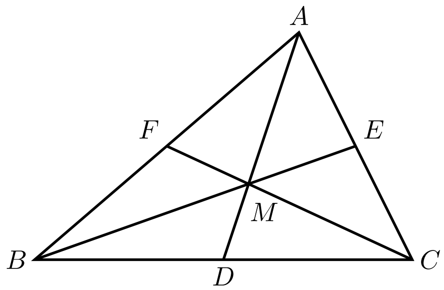
Problem 4. Find the path of minimal length that bisects the area of an equilateral triangle whose edge length is 1.

Problem 5. (Leon Bankoff, *Crux Mathematicorum*, March 1980) Consider the following sequence of circles.



Circles C_0 and C_1 have radius 1 and are tangent to each other and to the horizontal line ℓ. Circle C_2 is tangent to both of these circles and to ℓ. Continue drawing circles in this way so that circle C_n is tangent to circle C_0, circle C_{n-1}, and to ℓ. Find the radius of circle C_n.

Problem 6. The common point of intersection of the medians of a triangle is called the *centroid*. Suppose that triangle ABC has centroid M.

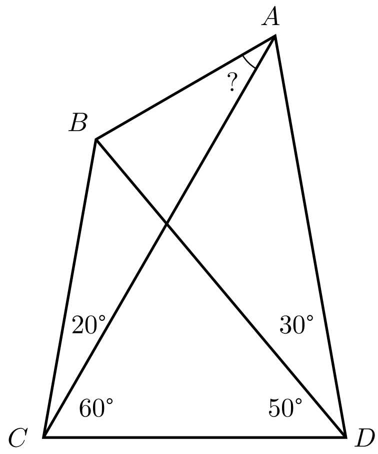


(a) Show that segments into which the centroid divides each median have lengths in a 2 : 1 ratio; for example, $AM = 2 \cdot DM$.

(b) Show that

$$(AM)^2 + (BM)^2 + (CM)^2 = \frac{(AB)^2 + (AC)^2 + (BC)^2}{3}.$$

Problem 7. (Langley's Adventitious Angles) Find the measure of $\angle BAC$ in quadrilateral $ABCD$.

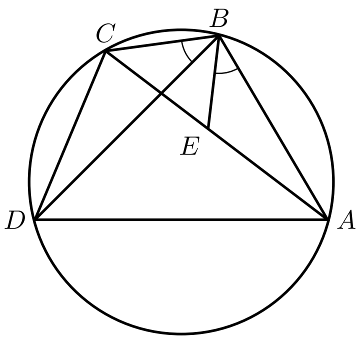


Problem 8. A *cyclic* quadrilateral is a quadrilateral that can be inscribed in a circle.

(a) Show that a quadrilateral is cyclic if and only if opposite angles sum to 180°.

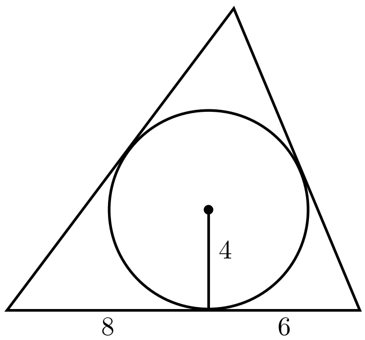
(b) Prove Ptolemy's Theorem: In a cyclic quadrilateral, the product of the diagonals is equal to the sum of the products of opposite sides.

$$AC \cdot BD = AB \cdot CD + AD \cdot BC.$$



Hint: construct the point E so that $\angle ABE \cong \angle CBD$.

Problem 9. The inscribed circle of a triangle has radius 4, and its point of tangency with one of the sides of the triangle divides it into segments of lengths 6 and 8. Find the lengths of the other two sides.



Problem 10. Let $p, q, r \in \mathbb{C}$, and suppose that the complex roots of

$$z^3 - 3pz^2 + 3qz - r = 0$$

are distinct and non-collinear. Consider the roots as the vertices of a triangle in the complex plane.

(a) Show that p is the centroid of this triangle.

(b) Show that the triangle is equilateral if and only if $p^2 = q$.

Problem 11. The *circumcenter* of a triangle is the center of the circumscribed circle; it can be constructed as the common intersection point of the perpendicular bisectors of the sides. The *orthocenter* of a triangle is the common intersection point of the altitudes. Suppose that triangle ABC has centroid M, circumcenter C, and orthocenter O.

(a) Show that M, O, and C are collinear.

(b) Show that $MO = 2 \cdot MC$.

Problem 12. A *lattice point* is a point in the xy-plane whose coordinates are integers. Show that the square is the only regular polygon that can be drawn in the plane so that all of its vertices are lattice points.