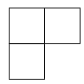
# Induction

Ksenija Simic-Muller and Matt Ondrus

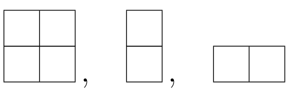
Feb 28, 2007

**Example 1.** Every positive integer $n$ can be expressed as $n = c_0 + c_1 2^1 + c_2 2^2 + \cdots + c_M 2^M$ for some $M \geq 0$, where $c_i \in \{0, 1\}$ for all $i$.

**Example 2.** (**Note:** This is an example of a *wrong* proof.)
Suppose that $S : \mathbb{Z}_{\geq 1} \to \mathbb{Z}$ is a function with the property that $S(n) = 5S(n-1) - 6S(n-2)$, where $S(1) = 9$ and $S(2) = 20$. Prove that $S(n) = 3 \cdot 2^n + 3^n$.

**Example 3.** Suppose you are given a square *checkerboard* with side-length $2^n$ and with one missing square. Prove that the remaining squares on the board can be tiled with trominos. Note that a tromino is an object shaped like  .

**Problem 1.** A winning configuration in the game of MiniTetris is a complete tiling of a $2 \times n$ board using only the three shapes shown below:



Prove that the number of winning configurations on a $2 \times n$ MiniTetris board ($n \geq 1$) is $T_n = (2^{n+1} + (-1)^n)/3$.

**Problem 2.** We are given a chocolate bar with $m \times n$ squares of chocolate, and our task is to divide it into $mn$ individual squares. We are only allowed to split one piece of chocolate at a time using a vertical or a horizontal break. For example, suppose that the chocolate bar is $2 \times 2$. The first split makes two pieces, both $2 \times 1$. Each of these pieces requires one more split to form single squares. This gives a total of three splits.
Use strong induction to conclude the following: Theorem. To divide up a chocolate bar with $m \times n$ squares, we need $mn - 1$ splits.

**Problem 3.** You begin with a stack of $n$ boxes. Then you make a sequence of moves. In each move, you divide one stack of boxes into two nonempty stacks. The game ends when you have $n$ stacks, each containing a single box. You earn points for each move; in particular, if you divide one stack of height $a + b$ into two stacks with heights $a$ and $b$, then you score $ab$ points for that move. Your overall score is the sum of the points that you earn for each move. What strategy should you use to maximize your total score?

**Problem 4.** Prove that consecutive Fibonacci numbers are always relatively prime.

**Problem 5.** Show that every positive integer can be expressed uniquely as the sum of distinct, non-consecutive Fibonacci numbers (here, non-consecutive means that no two of the Fibonacci numbers in the sum are consecutive Fibonacci numbers).

**Problem 6.** Let $n = 2^k$. Prove that we can select $n$ integers from any $(2n-1)$ integers such that their sum is divisible by $n$.

**Problem 7.** Prove that if you triangulate a convex $n$-gon, then there are at least two vertices of degree two. Note: Think of a convex $n$-gon as a graph consisting of $n$ vertices and $n$-edges arranged in a cycle. To *triangulate* an $n$-gon is to join non-adjacent vertices with edges in such a way that no edges cross each other and all of the resulting faces are triangles.

**Problem 8.** Let $S(n)$ denote the number of strings of length $n$ built from the alphabet $\{H, T\}$ that do not contain the substring $HH$. Prove that

$$S(n) = \left(\frac{5+3\sqrt{5}}{10}\right)\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{5-3\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

**Problem 9.** Prove that the faces of a planar graph can be colored with two colors (so that no two adjacent faces are the same color) iff all of its vertices have even degree.

**Problem 10.** In an $m \times n$ matrix of real numbers, we mark at least $p$ of the largest numbers ($p \le m$) in every column, and at least $q$ of the largest numbers ($q \le n$) in every row. Prove that at least $pq$ numbers are marked twice.

**Problem 11.** (Putnam, 1972) Show that, for all $n > 1$, $n$ does not divide $2^n - 1$.

**Problem 12.** There are no positive integer solutions of $x^4 + y^4 = z^2$.
Hint: You need to know that every positive integer solution of $a^2 + b^2 = c^2$ where $a, b$, and $c$ are relatively prime can be expressed in terms of two relatively prime numbers $m$, and $n$ where $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$.