Inequalities*

Math 294A: Problem Solving Seminar – Vera Furst

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- *Arithmetic-mean–geometric-mean inequality:* For $x_1, \ldots, x_n > 0$,

$$(x_1 x_2 \cdots x_n)^{1/n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}$$

  with equality if and only if $x_1 = x_2 = \cdots = x_n$.

- *Cauchy-Schwarz inequality:* For all real $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$,

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^2 \right)^{1/2} \left(\sum_{i=1}^{n} b_i^2 \right)^{1/2},$$

  with equality if and only if $a_1/b_1 = a_2/b_2 = \cdots = a_n/b_n \geq 0$. In vector form:

$$\vec{a} \cdot \vec{b} \leq \|\vec{a}\| \|\vec{b}\|,$$

  with equality if and only if $\vec{a} = c\vec{b}$ for some nonnegative scalar c.

Example 1. (Bernoulli's inequality) Prove that for $0 < a < 1$,

$$(1 + x)^a \leq 1 + ax$$

for all $x \geq -1$. How would the inequality change if $a > 1$ or $a < 0$?

Example 2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a twice differentiable function with $f''(x) \geq 0$ for all x. Prove that for all $a < b$,

$$f\left(\frac{a+b}{2}\right) \leq \frac{f(a) + f(b)}{2}.$$

Deduce the arithmetic-mean–quadratic-mean inequality for two numbers.

Example 3. If a, b, c, d are positive numbers such that $c^2 + d^2 = (a^2 + b^2)^3$, prove that

$$\frac{a^3}{c} + \frac{b^3}{d} \geq 1,$$

with equality if and only if $ad = bc$.

Problem 1. Let $a,\ b,\ c$ be the sides of a triangle. Show that

$$ab + bc + ca \leq a^2 + b^2 + c^2 \leq 2(ab + bc + ca).$$

*These problems (or some version of them) all appear either on previous Putnam exams or in the books *Problem-Solving Through Problems* by L.C. Larson and *Problem-Solving Strategies* by A. Engel.

Problem 2. A farmer with 1000 feet of fencing wishes to fence a rectangular field adjacent to a straight river. Naturally, no fence is needed along the river. Use the arithmetic-mean–geometric-mean inequality to find the dimensions of the field that maximize its area.

Problem 3. Prove that for each positive integer n,

$$\left(1+\frac{1}{n}\right)^n < \left(1+\frac{1}{n+1}\right)^{n+1}.$$

We proved this inequality last semester using the binomial theorem; this time, do it by showing that $f(x) = (1 + 1/x)^x$ is an increasing function.

Problem 4. (Jensen's inequality) Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function with $f''(x) \geq 0$ for all x. If $\lambda_1, \dots, \lambda_n$ are positive real numbers such that $\lambda_1 + \cdots + \lambda_n = 1$, then

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i f(x_i)$$

for any $x_1, \dots, x_n \in \mathbb{R}$. Deduce the general arithmetic-mean–geometric-mean inequality.

Problem 5. For which real numbers k does the inequality $\cosh x \leq e^{kx^2}$ hold for all real x?

Problem 6. Show that if a, b, c are positive numbers with $a + b + c = 1$, then

$$\left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2 + \left(c+\frac{1}{c}\right)^2 \geq \frac{100}{3}.$$

Problem 7. Show that if $C_k = \binom{n}{k}$ for $n > 2$ and $1 \leq k \leq n$, then

$$\sum_{k=1}^{n} \sqrt{C_k} \leq \sqrt{n(2^n - 1)}.$$

Problem 8. Show that for any integer $n > 1$,

$$\frac{1}{e} - \frac{1}{ne} < \left(1-\frac{1}{n}\right)^n < \frac{1}{e} - \frac{1}{2ne}.$$

Problem 9. Given $n \geq 2$, which of the two numbers is larger:

(a) An exponential tower of n 2's or an exponential tower of $(n-1)$ 3's?

(b) An exponential tower of n 3's or an exponential tower of $(n-1)$ 4's?

Problem 10. Let $x, y > 0$, and let s be the smallest of the numbers $x,\ y + 1/x,\ 1/y$. Find the greatest possible value of s. For which x, y is this value assumed?

Problem 11. Suppose twenty disjoined squares lie inside a square of side 1. Prove that there are four squares among them such that the sum of the lengths of their sides does not exceed $2/\sqrt{5}$.

Problem 12. Find all positive integers n such that

$$3^n + 4^n + \cdots + (n+2)^n = (n+3)^n.$$