$\zeta(3)$

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Contents

1 Introduction

1.1 The problem

For integers $k \geq 2$, the Riemann zeta function is defined by

$$\zeta(k) = \sum_{n \geq 1} \frac{1}{n^k}$$

The convergence of the above sum is standard, and can be showed using the efficient integral criterion or the elegant Cauchy criterion. The central question of this paper is the irrationality of $\zeta(3)$, viewed in the general context of determining the arithmetic nature (rational vs. irrational, algebraic vs. transcendental) of the $\zeta(k),\ \ k \geq 2$.

For even k, the answer turns out to be fairly simple: $\zeta(k)$ is a rational multiple of π^k, hence transcendental. For odd k, we know almost nothing. Apéry broke the ice in 1978 with a miraculous proof that $\zeta(3)$ is irrational[1] but whether $\zeta(3)$ is transcendent or whether $\zeta(3)$ is a rational multiple of π^3, is still unknown. So far, nor Apéry's proof, nor subsequent proofs of the irrationality of $\zeta(3)$ have not been extended to yield similar results for the values of the zeta function at odd integers greater than 3.

Recently, Ball and Rivoal showed that infinitely many of the $\zeta(k)$'s, where k runs over the odd integers, are irrational. The last piece of news seems to be Zudilin's result that one of the $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational.

1.2 The strategy

The elegant and magic-free proof of the irrationality of $\zeta(3)$ that we give in what follows is due to Beukers [3]. Just as Apéry's proof, Beukers' proof deals in parallel with both $\zeta(2)$ and $\zeta(3)$. Of course, nothing new is learnt about $\zeta(2)$, but it is a good preamble to proving the irrationality of $\zeta(3)$ which is slightly more involved. Once again, the two proofs follow the same exact lines (I'm tempted to call them iso-proofs).

The basic idea of the proofs is quite simple: let ξ be the real number we want to stigmatize as being irrational. If we have that

$$0 < |a_n + b_n \xi| < \alpha^n$$

for some $0 < \alpha < 1$, a_n, b_n integers and n sufficiently large, then it follows immediately that ξ is irrational. Else, if $\xi = p/q$ then the closest an expression of the form $a_n + b_n \xi$ $(a_n, b_n \in \mathbb{Z})$ could come to 0 without vanishing would be $1/q$ i.e $a_n + b_n \xi$ would be bounded away from zero

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[1] Check [5] for a lively description of Apéry's proof. Read and wonder.

independently of n.[2]

Before proceeding to the proofs, let us sharpen our tools. First, the Legendre-type polynomials

$$P_n(x) = \frac{1}{n!}\frac{d^n}{dx^n}(x^n(1-x)^n) \quad (n \geq 0)$$

which, using the binomial formula for differentiation, are easely seen to have integer coefficients. These polynomials are particularly easy to manipulate when performing integration by parts. Using the binomial expansion again, we obtain that for $i \leq n-1$

$$\frac{d^i}{dx^i}(x^n(1-x)^n)(0) = \frac{d^i}{dx^i}(x^n(1-x)^n)(1) = 0$$

so for integrable functions g we can write

$$\begin{aligned}
\int_0^1 P_n(x)g(x)dx &= \int_0^1 \frac{1}{n!}\frac{d^n}{dx^n}(x^n(1-x)^n)g(x)dx \\
&= \left[\frac{1}{n!}\frac{d^{n-1}}{dx^{n-1}}(x^n(1-x)^n)g(x)\right]_0^1 - \int_0^1 \frac{1}{n!}\frac{d^{n-1}}{dx^{n-1}}(x^n(1-x)^n)g'(x)dx \\
&= -\int_0^1 \frac{1}{n!}\frac{d^{n-1}}{dx^{n-1}}(x^n(1-x)^n)g'(x)dx
\end{aligned}$$

Repeating this process n times and taking the absolute value we obtain

$$\left|\int_0^1 P_n(x)g(x)dx\right| = \left|\int_0^1 \frac{1}{n!}x^n(1-x)^n g^{(n)}(x)dx\right| \tag{1}$$

Second, the least common multiple of 1, 2, ..., n:

$$d_n = [1, 2, \ldots, n]$$

whose significant feature is that $d_n < 3^n$ for n sufficiently large. To justify this estimate on d_n, notice that $d_n = \prod p^{\alpha_p}$, where the product is taken over all primes $p \leq n$ and each α_p is the greatest integer with $p^{\alpha_p} \leq n$. So $d_n \leq n^{\pi(n)}$, where $\pi(n)$ denotes the number of primes less than n. The prime number theorem says that

$$1 = \lim_{n \longrightarrow \infty} \frac{\pi(n)\ln n}{n} = \lim_{n \longrightarrow \infty} \frac{\ln n^{\pi(n)}}{n}$$

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[2] Consider $G = \{a + b\xi : a, b \in \mathbb{Z}\}$ as an additive subgroup of $\mathbb{R}$, so the above condition on ξ simply says that 0 is an accumulation point for G. It is folklore that the following assertions about a subgroup $0 \neq G \subseteq \mathbb{R}$ are equivalent:

a) 0 is an accumulation point for G
b) G is a dense subgroup of $\mathbb{R}$
c) G is not cyclic i.e. G is not isomorphic to $\mathbb{Z}$

But G not cyclic is equivalent to ξ being irrational. What we said in so many words is actually the content of a result going back to Kronecker: $\{a + b\xi : a, b \in \mathbb{Z}\}$ is dense on the real line if and only if ξ is irrational.

so for n sufficiently large we have $\frac{\ln n^{\pi(n)}}{n} < \ln 3$ i.e. $d_n \leq n^{\pi(n)} < 3^n$.

As a last piece of notation, we will use $\int_\square$ to mean integral over the unit square $[0,1] \times [0,1]$. Whenever it occurs, the interchange of integral with infinite sum is essentially motivated by the monotone convergence theorem, the interchange of integral with derivative is motivated by the utter smoothness of the functions involved, whereas the constant use of improper integrals instead of limits of proper ones is somehow justified by the "uncertainty principle of writing proofs": a gain in rigor is a loss in clarity.

2 Morning warm-up: Irrationality of $\zeta(2)$

Lemma 2.1. *For all* $0 \leq x, y \leq 1$ *we have:*

$$\frac{x(1-x)y(1-y)}{1-xy} \leq \left(\frac{\sqrt{5}-1}{2}\right)^5$$

Proof. Let $f(x,y)$ be the function given in the lemma. Notice first that f vanishes on the boundary of $[0,1] \times [0,1]$. At $(1,1)$ the function f is not defined but we have $f(x,y) \longrightarrow 0$ as $x, y \nearrow 1$.

To find the maximum of f in the unit square, we solve the following system in $(0,1) \times (0,1)$:

$$\frac{\partial}{\partial x} f(x,y) = 0 = \frac{\partial}{\partial y} f(x,y)$$

which immediately takes the form:

$$1 - 2x + yx^2 = 0 = 1 - 2y + xy^2$$

Express y from the first relation, substitute in the second relation and obtain $x^3 - 2x + 1 = 0$, whose roots are $1, \frac{-1\pm\sqrt{5}}{2}$. Hence $x = \frac{\sqrt{5}-1}{2}$ and, by symmetry, $y = \frac{\sqrt{5}-1}{2}$, and that's where f achieves its maximum value $\left(\frac{\sqrt{5}-1}{2}\right)^5$. $\square$

Proposition 2.2. *Let* $r, s \in \mathbb{N}$. *Then*

$$\int_\square \frac{x^r y^s}{1-xy} dxdy \in \begin{cases} \zeta(2) + \frac{1}{d_r^2}\mathbb{Z} & \text{if } r = s \\ \frac{1}{d_r^2}\mathbb{Z} & \text{if } r > s \end{cases}$$

Proof. For any real $a \geq 0$ we have

$$\begin{aligned} \int_\square \frac{x^{r+a} y^{s+a}}{1-xy} dxdy &= \int_\square x^{r+a} y^{s+a} \sum_{n \geq 0} (xy)^n dxdy = \sum_{n\geq 0} \int_\square x^{n+r+a} y^{n+s+a} dxdy \\ &= \sum_{n \geq 0} \int_0^1 x^{n+r+a} dx \int_0^1 y^{n+s+a} dy = \sum_{n\geq 0} \frac{1}{n+r+a+1} \cdot \frac{1}{n+s+a+1} \qquad (2) \end{aligned}$$

For the present proof, we only use (2) at $a = 0$, but the full force of (2) will be needed later, when dealing with $\zeta(3)$. If $r = s$ then at $a = 0$ we get

$$\int_{\Box} \frac{x^r y^r}{1-xy} dxdy = \sum_{n \geq 0} \frac{1}{n+r+1} \cdot \frac{1}{n+r+1} = \zeta(2) - \frac{1}{1^2} - \cdots - \frac{1}{r^2} \in \zeta(2) + \frac{1}{d_r^2}\mathbb{Z}$$

In particular, for $r = 0$ we have the integral representation:

$$\zeta(2) = \int_{\Box} \frac{dxdy}{1-xy} \tag{3}$$

If $r > s$ then we can express the sum in (2) as

$$\begin{aligned} \sum_{n \geq 0} \frac{1}{n+r+a+1} \cdot \frac{1}{n+s+a+1} &= \frac{1}{r-s} \sum_{n \geq 0} \left(\frac{1}{n+s+a+1} - \frac{1}{n+r+a+1} \right) \\ &= \frac{1}{r-s} \left(\frac{1}{s+a+1} + \cdots + \frac{1}{r+a} \right) \end{aligned} \tag{4}$$

Set again $a = 0$ to get that the last sum can be expressed as a ratio whose denominator is d_r^2. □

Theorem 2.3. *$\zeta(2)$ is irrational.*

Proof. Consider the integral

$$\int_{\Box} \frac{P_n(x)(1-y)^n}{1-xy} dxdy$$

As P_n is a polynomial with integer coefficients, this integral equals $(a_n + b_n\zeta(2))/d_n^2$ $(a_n, b_n \in \mathbb{Z})$. On the other hand, using (1) we have

$$\begin{aligned} \left| \int_{\Box} \frac{P_n(x)(1-y)^n}{1-xy} dxdy \right| &= \left| \int_0^1 P_n(x) \left(\int_0^1 \frac{(1-y)^n}{1-xy} dy \right) dx \right| \\ &= \left| \int_0^1 \frac{x^n(1-x)^n}{n!} \frac{d^n}{dx^n} \left(\int_0^1 \frac{(1-y)^n}{1-xy} dy \right) dx \right| \\ &= \left| \int_0^1 \frac{x^n(1-x)^n}{n!} \left(\int_0^1 \frac{d^n}{dx^n} \left(\frac{(1-y)^n}{1-xy} \right) dy \right) dx \right| \\ &= \left| \int_0^1 \frac{x^n(1-x)^n}{n!} \left(\int_0^1 \frac{n! y^n (1-y)^n}{(1-xy)^{n+1}} dy \right) dx \right| \\ &= \int_{\Box} \frac{x^n(1-x)^n y^n (1-y)^n}{(1-xy)^{n+1}} dxdy \end{aligned}$$

Hence the integral we considered does not vanish and by lemma 2.1 we have:

$$0 < \left| \frac{a_n + b_n\zeta(2)}{d_n^2} \right| \leq \left(\frac{\sqrt{5}-1}{2} \right)^{5n} \int_\square \frac{dxdy}{1-xy} = \left(\frac{\sqrt{5}-1}{2} \right)^{5n} \zeta(2)$$

Using the estimate on d_n we can write for n sufficiently large:

$$0 < |a_n + b_n\zeta(2)| < 9^n \left(\frac{\sqrt{5}-1}{2} \right)^{5n} \zeta(2) < 0.9^n$$

which implies that $\zeta(2)$ is irrational. □

We can actually compute $\zeta(2)$ by using a double-integral representation.[3] It's not really (3) that we need, but a slightly modified formula:

$$\int_\square \frac{dxdy}{1-x^2y^2} = \int_\square \sum_{n\geq 0} (xy)^{2n} dxdy = \sum_{n\geq 0} \left(\frac{1}{2n+1} \right)^2 = \frac{3}{4}\zeta(2)$$

To compute the above double integral, we will use a clever substitution:

$$A = \{u, v : u \geq 0,\ v \geq 0,\ u+v \leq \frac{\pi}{2}\} \longrightarrow [0,1] \times [0,1], \quad (u,v) \longrightarrow \left(\frac{\sin u}{\cos v}, \frac{\sin v}{\cos u} \right)$$

Some trigonometric gymnastics show that the above transformation is well-defined and it is bijective, its inverse being in fact given by

$$(x,y) \longrightarrow \left(\arctan x\sqrt{\frac{1-y^2}{1-x^2}}, \arctan y\sqrt{\frac{1-x^2}{1-y^2}} \right)$$

The Jacobian of the transformation is:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \cos u/\cos v & \sin u \sin v/\cos^2 v \\ \sin u \sin v/\cos^2 u & \cos v/\cos u \end{vmatrix} = 1 - \frac{\sin^2 u \sin^2 v}{\cos^2 u \cos^2 v} = 1 - x^2y^2$$

We immediately get

$$\frac{3}{4}\zeta(2) = \int_\square \frac{dxdy}{1-x^2y^2} = \int_A dudv = Area(A) = \frac{\pi^2}{8} \iff \zeta(2) = \frac{\pi^2}{6}$$

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[3] Enjoy Chapman's survey "Evaluating $\zeta(2)$" containing 14 different ways of proving that $\zeta(2) = \pi^2/6$. Available at www.maths.ex.ac.uk/ rjc/etc

3 Noon climax: Irrationality of $\zeta(3)$

Lemma 3.1. *For all* $0 \leq x, y, w \leq 1$ *we have:*

$$\frac{x(1-x)y(1-y)w(1-w)}{1-(1-xy)w} \leq (\sqrt{2}-1)^4$$

Proof. Let $f(x, y, w)$ be the function given in the lemma. Notice that f vanishes on the boundary of the unit cube $[0,1] \times [0,1] \times [0,1]$. On the edges $x = 0, w = 1$ and $y = 0, w = 1$ the function f is not defined but we have $f(x, y, w) \longrightarrow 0$ as $x \searrow 0, w \nearrow 1$ or as $y \searrow 0, w \nearrow 1$.

To find the maximum of f in the unit cube, we solve the following system in $(0,1)\times(0,1)\times(0,1)$:

$$\frac{\partial}{\partial x} f(x, y, w) = \frac{\partial}{\partial y} f(x, y, w) = \frac{\partial}{\partial w} f(x, y, w) = 0$$

which immediately takes the form:

$$1 - 2w + (1-xy)w^2 = (1-2x) - (1-2x+x^2y)w = (1-2y) - (1-2y+xy^2)w = 0$$

Equating the expressions for w from the last two relations gives $x = y$. Use this in the first relation and get $w = \frac{1}{1+x}$. The last two relations, now rendered equivalent, give $w = \frac{1-2x}{1-2x+x^3}$. Equating these two expressions quickly leads to $x^2 + 2x - 1 = 0$ whose roots are $-1 \pm \sqrt{2}$. Hence $x = y = \sqrt{2} - 1$ and $w = 1/\sqrt{2}$ and that's where f achieves its maximum value $(\sqrt{2}-1)^4$. $\square$

Proposition 3.2. *Let* $r, s \in \mathbb{N}$. *Then*

$$\int_\square -\frac{x^r y^s \ln xy}{1-xy} dxdy \in \begin{cases} 2\zeta(3) + \frac{1}{d_r^3}\mathbb{Z} & \text{if } r = s \\ \frac{1}{d_r^3}\mathbb{Z} & \text{if } r > s \end{cases}$$

Proof. If $r = s$, it follows by (2) that for all $a \geq 0$:

$$\int_\square \frac{x^{r+a} y^{r+a}}{1-xy} dxdy = \sum_{n \geq 0} \frac{1}{(n+r+a+1)^2}$$

Differentiate with respect to a to obtain:

$$\int_\square \frac{x^{r+a} y^{r+a} \ln xy}{1-xy} dxdy = -2 \sum_{n \geq 0} \frac{1}{(n+r+a+1)^3}$$

At $a = 0$ this says

$$\int_\square \frac{x^r y^r \ln xy}{1-xy} dxdy = -2 \sum_{n \geq 0} \frac{1}{(n+r+1)^3} = -2 \left(\zeta(3) - \frac{1}{1^3} - \cdots - \frac{1}{r^3} \right) \in -2\zeta(3) - \frac{1}{d_r^3}\mathbb{Z}$$

In particular, for $r = 0$ we have the integral representation:

$$\zeta(3) = -\frac{1}{2}\int_{\square} \frac{\ln xy}{1-xy}dxdy$$

If $r > s$, remember that relation (4) says

$$\int_{\square} \frac{x^{r+a}y^{s+a}}{1-xy}dxdy = \frac{1}{r-s}\left(\frac{1}{s+a+1} + \cdots + \frac{1}{r+a}\right)$$

Differentiate with respect to a and obtain:

$$\int_{\square} \frac{x^{r+a}y^{s+a}\ln xy}{1-xy}dxdy = \frac{-1}{r-s}\left(\frac{1}{(s+a+1)^2} + \cdots + \frac{1}{(r+a)^2}\right)$$

For $a = 0$ we get

$$\int_{\square} \frac{x^{r}y^{s}\ln xy}{1-xy}dxdy = \frac{-1}{r-s}\left(\frac{1}{(s+1)^2} + \cdots + \frac{1}{r^2}\right) \in \frac{1}{d_r^3}\mathbb{Z}$$

□

Theorem 3.3. $\zeta(3)$ *is irrational.*

Proof. Consider the integral

$$\int_{\square} -\frac{P_n(x)P_n(y)\ln xy}{1-xy}dxdy$$

Since $P_n \in \mathbb{Z}[X]$, by previous proposition this integral equals $(a_n + b_n\zeta(3))/d_n^3$ with a_n, b_n integers. On the other hand, since

$$-\frac{\ln xy}{1-xy} = \int_0^1 \frac{1}{1-(1-xy)z}dz$$

we have

$$\begin{aligned}
\left|\int_{\square} -\frac{P_n(x)P_n(y)\ln xy}{1-xy}dxdy\right| &= \left|\int_0^1 P_n(x)\left(\int_{\square}\frac{P_n(y)}{1-(1-xy)z}dydz\right)dx\right| \\
&= \left|\int_0^1 \frac{x^n(1-x)^n}{n!}\frac{d^n}{dx^n}\left(\int_{\square}\frac{P_n(y)}{1-(1-xy)z}dydz\right)dx\right| \\
&= \left|\int_0^1 \frac{x^n(1-x)^n}{n!}\left(\int_{\square}\frac{d^n}{dx^n}\left(\frac{P_n(y)}{1-(1-xy)z}\right)dydz\right)dx\right| \\
&= \left|\int_0^1 \frac{x^n(1-x)^n}{n!}\left(\int_{\square}\frac{(-1)^n n! P_n(y)y^n z^n}{(1-(1-xy)z)^{n+1}}dydz\right)dx\right| \\
&= \left|\int_0^1 P_n(y)\left(\int_{\square}\frac{x^n(1-x)^n y^n z^n}{(1-(1-xy)z)^{n+1}}dxdz\right)dy\right|
\end{aligned}$$

Make the change of variables $w = \frac{1-z}{1-(1-xy)z}$ (an involution) so $dw = \frac{-xy}{(1-(1-xy)z)^2}dz$ and we get:

$$\begin{aligned}
\left|\int_\Box -\frac{P_n(x)P_n(y)\ln xy}{1-xy}dxdy\right| &= \left|\int_0^1 P_n(y)\left(\int_\Box \frac{(1-x)^n(1-w)^n}{1-(1-xy)w}dxdw\right)dy\right| \\
&= \left|\int_0^1 \frac{y^n(1-y)^n}{n!}\frac{d^n}{dy^n}\left(\int_\Box \frac{(1-x)^n(1-w)^n}{1-(1-xy)w}dxdw\right)dy\right| \\
&= \left|\int_0^1 \frac{y^n(1-y)^n}{n!}\left(\int_\Box \frac{d^n}{dy^n}\left(\frac{(1-x)^n(1-w)^n}{1-(1-xy)w}\right)dxdw\right)dy\right| \\
&= \left|\int_0^1 \frac{y^n(1-y)^n}{n!}\left(\int_\Box \frac{(-1)^n n!(1-x)^n(1-w)^n x^n w^n}{(1-(1-xy)w)^{n+1}}dxdw\right)dy\right| \\
&= \int\int\int_0^1 \frac{x^n(1-x)^n y^n(1-y)^n w^n(1-w)^n}{(1-(1-xy)w)^{n+1}}dxdydw
\end{aligned}$$

Thus the integral we considered does not vanish and by lemma 3.1 we have:

$$0 < \left|\frac{a_n + b_n\zeta(3)}{d_n^3}\right| \le (\sqrt{2}-1)^{4n}\int\int\int_0^1 \frac{dxdydw}{1-(1-xy)w} = (\sqrt{2}-1)^{4n}\int_\Box -\frac{\ln xy}{1-xy}dxdy$$

That is

$$0 < |a_n + b_n\zeta(3)| \le d_n^3(\sqrt{2}-1)^{4n}2\zeta(3)$$

By our estimate on d_n we can write for n sufficiently large:

$$0 < |a_n + b_n\zeta(3)| < 27^n(\sqrt{2}-1)^{4n}2\zeta(3) < 0.8^n$$

which implies that $\zeta(3)$ is irrational. $\Box$

4 Evening contemplation

4.1 The zeta function at even integers

We outline in this section the computation of $\zeta(2k)$. There are two notions that we'll need on the way: the Bernoulli numbers and the gamma function.

The Bernoulli numbers $(B_n)_{n\ge 0}$ are defined by the following power expansion around 0:

$$\frac{x}{e^x-1} = \sum_{n\ge 0} B_n\frac{x^n}{n!}$$

They can be computed using the recursion formula [4]

$$\binom{n+1}{1}B_n + \binom{n+1}{2}B_{n-1} + \cdots + \binom{n+1}{n}B_1 + B_0 = 0$$

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[4] Follows by considering the coefficient of x^n in: $1 = \left(\frac{x}{e^x-1}\right)\left(\frac{e^x-1}{x}\right) = \left(\sum_{n\ge 0} B_n\frac{x^n}{n!}\right)\left(\sum_{n\ge 0}\frac{x^n}{(n+1)!}\right)$.

In particular, the Bernoulli numbers are rational numbers; explicitly, they start off as 1,$-\frac{1}{2}$,$\frac{1}{6}$,0, $-\frac{1}{30}$,0,$\frac{1}{42}$,0,$-\frac{1}{30}$,0, $\frac{5}{66}$,.... And numbers don't lie: $B_n = 0$ for odd n, $n \neq 1$.[5]

The gamma function is defined for $\Re(s) > 0$ as

$$\Gamma(s) = \int_0^\infty e^{-u} u^{s-1} du$$

and then extended to a function that is analytic in the whole complex plane, except simple poles at 0, -1, -2, -3,.... Integration by parts easily shows that $\Gamma(n+1) = n!$.

The Riemann zeta function is first defined on the open half-plane $\Re(s) > 1$ (where the series converges absolutely) by

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

Using contour integration, one provides the analytic continuation of the zeta function to the whole complex plane, so that the zeta function is now analytic everywhere except for a simple pole at $s = 1$ with residue 1 i.e. $\lim_{s \to 1}(s-1)\zeta(s) = 1$. Thus extended, the zeta function satisfies the Riemann functional equation relating the values of ζ on arguments symmetric about the critical line $\Re(s) = 1/2$:

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi s}{2}\right)\zeta(s)$$

and furthermore, the zeta function takes a simple form on negative integers:

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}$$

We obtain, in particular, the trivial zeros of the zeta function: $\zeta(-n) = 0$ for even $n \geq 2$. On the other hand, set $s = 2k$ (k a positive integer) in the functional equation for ζ to get

$$-\frac{B_{2k}}{2k} = \zeta(1-2k) = 2(2\pi)^{-2k}\Gamma(2k)\cos(\pi k)\zeta(2k) = 2(2\pi)^{-2k}(2k-1)!(-1)^k\zeta(2k)$$

that is

$$\zeta(2k) = \frac{(-1)^{k+1}B_{2k}}{2(2k)!}(2\pi)^{2k} \tag{5}$$

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[5] Indeed: $\frac{x}{e^x-1} + \frac{x}{2} = \frac{x}{2} \cdot \frac{e^{x/2}+e^{-x/2}}{e^{x/2}-e^{-x/2}}$ which is an even function, so $B_1 = -1/2$ and $B_n = 0$ for odd $n \geq 3$. On the other hand, for even n an interesting arithmetic information on the B_n's is given by the Clausen-von Staudt theorem: the denominator of B_n is $\prod_{\substack{p-1|n \\ p \text{ prime}}} p$.

Since π is transcendental, it follows that any integer power of π is transcendental, therefore $\zeta(2k)$ is transcendental.[6]

Instead of this long walk through the woods of the zeta function, one may choose a shortcut. Remember, our present aim is just the arithmetic nature of ζ at even integers.

Using elementary series manipulation, one can prove the following convolution formula for the $\zeta(2k)$'s (see [6] for the one-page proof)

$$(k + \frac{1}{2})\zeta(2k) = \sum_{i=1}^{k-1} \zeta(2i)\zeta(2k-2i) \quad (k \geq 2) \tag{6}$$

Visibly, this gives that $\zeta(2k)$ is a rational multiple of $\zeta(2)^k$. While (6) says nothing about $\zeta(2)$, we have already established that $\zeta(2) = \pi^2/6$. Hence $\zeta(2k)$ is a rational multiple of π^{2k}, which implies the transcendence of $\zeta(2k)$.

We can even use (6) to establish (5) in a very simple fashion. We first need the following:

Lemma 4.1. *Let* $C_{2k} = \frac{B_{2k}}{(2k)!}$. *Then:* $-(2k+1)C_{2k} = \sum_{i=1}^{k-1} C_{2i}C_{2k-2i}$ *for* $k \geq 2$.

Proof. We have the following series expansion:

$$\frac{x}{e^x - 1} = -\frac{x}{2} + \sum_{k \geq 0} C_{2k}x^{2k} \tag{7}$$

We will compute in two ways the coefficient of x^{2k} for $k \geq 2$ in the expansion of $x^2/(e^x-1)^2$. On one hand, squaring (7) we obtain that the respective coefficient is $C_0C_{2k} + C_2C_{2k-2} + \cdots + C_{2k-2}C_2 + C_{2k}C_0$. On the other hand, differentiate (7) term by term to obtain:

$$\frac{1}{e^x-1} - \frac{x}{(e^x-1)^2} - \frac{x}{e^x-1} = -\frac{1}{2} + \sum_{k \geq 1} 2kC_{2k}x^{2k-1}$$

Multiply by x and use (7) to obtain:

$$\left(-\frac{x}{2} + \sum_{k \geq 0} C_{2k}x^{2k}\right) - \frac{x^2}{(e^x-1)^2} - x\left(-\frac{x}{2} + \sum_{k \geq 0} C_{2k}x^{2k}\right) = -\frac{x}{2} + \sum_{k \geq 1} 2kC_{2k}x^{2k}$$

hence the coefficient of x^{2k} for $k \geq 2$ in the expansion of $x^2/(e^x-1)^2$ is $(1-2k)C_{2k}$. Therefore $(1-2k)C_{2k} = C_0C_{2k} + C_2C_{2k-2} + \cdots + C_{2k-2}C_2 + C_{2k}C_0$ and, since $C_0 = 1$, we get the required relation. It terms of Bernoulli numbers this gives the following recurrence

$$-(2k+1)B_{2k} = \sum_{i=1}^{k-1} \binom{2k}{2i} B_{2i}B_{2k-2i}$$

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[6] Notice that no similar information about $\zeta(2k+1)$ can be obtained by plugging in $s = 2k+1$, since both sides of the functional equation will vanish.

which provides yet another way of proving that Bernoulli numbers are rational numbers once we establish that B_2 is rational. □

Using this lemma, relation (6) and the fact that $\zeta(2) = \pi^2/6$, it is an easy matter to establish by induction that $\zeta(2k) = (-1)^{k+1} C_{2k} (2\pi)^{2k}/2$ which is just (5).

4.2 A qualitative result on the zeta function at odd integers

Very recently, Ball and Rivoal have proved the following:

Theorem 4.2. *Let* $n \geq 3$ *be an odd integer and denote by* $\delta(n)$ *the dimension of the* $\mathbb{Q}$*-vector space generated by* $1, \zeta(3), \zeta(5), \ldots, \zeta(n)$*. Then* $\delta(n) \geq \frac{1}{3} \ln n$*.*

This may be quite far from the true asymptotic behavior of $\delta(n)$ since, for one thing, it does not yield the irrationality of $\zeta(3)$.[7] In fact, the above result does not indicate the irrationality of any specific $\zeta(2k+1)$. However, it is powerful enough to give that there is an infinite subset of $\{1, \zeta(3), \zeta(5), \zeta(7), \ldots\}$ that is linearly independent over $\mathbb{Q}$.[8] In particular, at most one element of the subset is rational, hence infinitely many of the values of the zeta function at odd integers are irrational.

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[7] Saying that $\zeta(3)$ is irrational is equivalent to saying that $\delta(3) \geq 2$; the theorem only gives $\delta(3) \geq \frac{1}{3} \ln 3 \approx 0.366$.

[8] This fits the hypothesis, still far from our present knowledge, that for k odd $\zeta(k)$ is a rational multiple of π^k.