

Topics in Number Theory : p-adic numbers, p-adic analysis, and applications

McGill 189-726 A

Problem Set #9

Due Monday, December 3, 2001

1. We give an example, due to Koblitz, of a series in \mathbb{Q} which converges to an element of \mathbb{Q} in both of \mathbb{R} and some \mathbb{Q}_p , but converges to different values in the two spaces. Let $x_n = \sum_{k=0}^n \binom{1/2}{k} \left(\frac{7}{9}\right)^k$, so that the x_n are the partial sums for the binomial expansion of $(1 + \frac{7}{9})^{1/2}$. Show that $x_n \rightarrow 4/3$ in \mathbb{R} and that $x_n \rightarrow -4/3$ in \mathbb{Q}_7 .
2. Show that there is a unique function $\log_p : \mathbb{C}_p \setminus \{0\} \rightarrow \mathbb{C}_p$ satisfying:
 - (a) if $|x - 1| < 1$ then $\log_p(x)$ is the usual p -adic logarithm,
 - (b) $\log_p(xy) = \log_p(x) + \log_p(y)$, and
 - (c) $\log_p(p) = 0$.
3. *The p-adic gamma function.* The complex gamma function has the special values $\Gamma(n) = (n-1)!$ for integers $s > 0$. The usual wisdom is that to be able to interpolate a p -adic gamma function, we should discard the terms divisible by p : that is, we should attempt to interpolate the function $\Gamma^*(n) = \prod_{k < n, p \nmid k} k$. This turns out not to be quite right. However, if we set

$$\Gamma_p(n) = (-1)^n \prod_{k < n, p \nmid k} k,$$

prove that Γ_p extends uniquely to a continuous function $\mathbb{Z}_p \rightarrow \mathbb{Z}_p$.

4. Show that the p -adic gamma function defined in the previous problem satisfies the following identities:
 - (a) $\Gamma_p(s+1)/\Gamma_p(s) = \begin{cases} -s & \text{if } s \in \mathbb{Z}_p^\times \\ -1 & \text{if } s \in p\mathbb{Z}_p. \end{cases}$
 - (b) $\Gamma_p(s)\Gamma_p(1-s) = (-1)^{s_0}$, where s_0 is the unique integer in $\{1, \dots, p\}$ such that $s_0 \equiv s \pmod{p}$, and $p \neq 2$.
 - (c) Set $s_1 = (s - s_0)/p$. Let m be any integer not divisible by p , and let $p \neq 2$. Then

$$\frac{\prod_{h=0}^{m-1} \Gamma_p\left(\frac{s+h}{m}\right)}{\Gamma_p(s) \prod_{h=1}^{m-1} \Gamma_p\left(\frac{h}{m}\right)} = m^{1-s_0} (m^{-(p-1)})^{s_1}.$$

Hint: in each case, note that it suffices to prove the identity for s a positive integer. (Why?) Then for parts (b) and (c), proceed by induction.

5. Verify that the measures μ_z on \mathbb{Z}_p that we defined in class are additive. Namely, check that if $a \in \{0, \dots, p^N - 1\}$, then

$$\mu_z(a + p^N \mathbb{Z}_p) = \sum_{b=0}^{p-1} \mu_z((a + bp^N) + p^{N+1} \mathbb{Z}_p)$$