

- Let  $f(x) = \sum_{k=0}^{\infty} \binom{1/2}{k} x^k$ , the series for  $(1+x)^{1/2}$ . We have seen on problem set #8 that the coefficients  $\binom{1/2}{k}$  are bounded in absolute value, and so  $f(x)$  converges when  $|x| < 1$ . We first will verify on this region that in fact  $f(x)^2 = 1+x$ . To see this, note that (rearranging the double-sum) we have

$$\left( \sum_{k=0}^{\infty} \binom{1/2}{k} x^k \right)^2 = \sum_{k=0}^{\infty} \sum_{j=0}^k \binom{1/2}{j} \binom{1/2}{k-j} x^k.$$

But  $\sum_{j=0}^k \binom{1/2}{j} \binom{1/2}{k-j} = \binom{1}{k}$  is 1 if  $k = 0, 1$  and is 0 otherwise. So indeed  $f(x)^2 = 1+x$  when  $|x| < 1$ .

The sum for  $f(7/9)$  converges in  $\mathbb{R}$ , and so the sequence  $x_n$  converges in  $\mathbb{R}$  to a limit  $L$  satisfying  $L^2 = 1 + 7/9$ . Hence  $L$  is either  $4/3$  or  $-4/3$ . Now, the zeroth term of the sum for  $f(7/9)$  is 1. Moreover, one checks that  $\left| \binom{1/2}{k} \right|_{\infty} \leq 1/2$  for  $k \geq 1$ , and so the sum of the remaining terms is bounded by  $\frac{1}{2} \frac{7/9}{1-7/9} = 7/4$ . Hence  $L \geq 1 - 7/4 > -4/3$ , and so  $L = 4/3$ .

Similarly, in  $\mathbb{Q}_7$  we have  $|7/9|_7 < 1$ , and so  $f(7/9)$  converges. Once again  $x_n$  tends to a limit  $L'$  which is either  $4/3$  or  $-4/3$ . This time, however, we see that the zeroth term of the sum for  $f(7/9)$  is 1, while the remaining terms all lie in  $7\mathbb{Z}_7$ . Hence  $L' - 1 \in 7\mathbb{Z}_7$ , forcing  $L' = -4/3$ .

- Let  $x \in \mathbb{C}_p^\times$ , let  $r = v_p(x) = \frac{m}{n}$ , and let  $y$  be any root of  $y^n - p^m = 0$ . Then  $x/y \in \mathfrak{O}_{\mathbb{C}_p}^\times$ . Suppose that  $x/y \mapsto \bar{\zeta}$  via the reduction map  $\mathfrak{O}_{\mathbb{C}_p}^\times \rightarrow \overline{\mathbb{F}}_p$ , and let  $\zeta$  be the Teichmüller lift of  $\bar{\zeta}$ , so that  $\zeta$  is a root of unity. Write  $x = y\zeta x_1$ . Observe that  $|x_1 - 1| < 1$ .

If the desired extension  $f$  of  $\log_p$  exists, then it must be the case that  $f(x) = f(y) + f(\zeta) + f(x_1)$ . But  $nf(y) = mf(p) = 0$  so  $f(y) = 0$ , and similarly  $f(\zeta) = 0$ , so  $f(x) = f(x_1) = \log_p(x_1)$ . This shows that if  $f$  exists, it is unique. To show that  $f$  exists, we simply define  $f(x) = \log_p(x_1)$ . We must verify that this  $f$  is well-defined. This follows because the only choice we made was  $y$ , and changing  $y$  simply changes  $x_1$  by a factor of a root of unity, which does not affect  $\log_p(x_1)$ . It is evident that  $f$  so-defined satisfies the three desired properties.

- We claim that for a positive integer  $n$  we have  $\Gamma_p(n + p^N) \equiv \Gamma_p(n) \pmod{p^N}$  for any  $N \geq 1$  (or  $N \geq 3$  if  $p = 2$ ). From the claim, it follows that  $\Gamma_p(n) \equiv \Gamma_p(m) \pmod{p^N}$  if  $n \equiv m \pmod{p^N}$  for  $N \geq 1$  (or  $N \geq 3$  if  $p = 2$ ). Therefore  $\Gamma_p$  is uniformly continuous on the positive integers, and extends to a function on all of  $\mathbb{Z}_p$ .

To prove the claim, note that

$$\Gamma_p(n + p^N) - \Gamma_p(n) = (-1)^n \Gamma_p(n) \left( \left( (-1)^{p^N} \prod_{n \leq k < p^N + n, p \nmid k} k \right) - 1 \right).$$

However  $\prod_{n \leq k < p^N + n, p \nmid k} k \equiv \prod_{1 \leq k < p^N, p \nmid k} k \pmod{p^N}$ . We claim that this is  $-1 \pmod{p^N}$  if  $p$  is odd or  $p = 2$  and  $N = 2$ , and is 1 if  $p = 2$  and  $N \neq 2$ ; it is immediate that this is what we need to complete the solution. To verify the claim, note that the product is simply the product of all elements in  $(\mathbb{Z}/p^N\mathbb{Z})^\times$ . Multiplicative inverses pair up, and the remaining product is simply the product of all square roots of 1. If  $p$  is odd or if  $p = 2$  and  $N = 2$ , the square roots of 1 are simply  $\pm 1$ . For  $p = 2$  and  $N > 2$ , the square roots of 1 are  $\pm 1$  and  $2^{N-1} \pm 1$ , and their product is 1.

- Note that if  $n$  is a positive integer, then  $\Gamma_p(n) \notin p\mathbb{Z}_p$ , and so  $\Gamma_p : \mathbb{Z}_p \rightarrow \mathbb{Z}_p \setminus p\mathbb{Z}_p$ . In particular,  $\Gamma_p(s)$  is never zero for  $s \in \mathbb{Z}_p$ . We see, therefore, that the quotient  $\Gamma_p(s+1)/\Gamma_p(s)$  is a continuous function; indeed each side of the identities we wish to establish are continuous. Therefore it suffices to check the identities on positive integers.

- (a) If  $s \in p\mathbb{Z}$  is a positive integer, then directly from the definition we see that  $\Gamma_p(s+1) = -\Gamma_p(s)$ . Similarly if  $s$  is a positive integer not divisible by  $p$ , then  $\Gamma_p(s+1) = (-s)\Gamma_p(s)$ . The desired identity follows.
- (b) From the definition, we have  $\Gamma_p(1) = -1$ . By part (a), we get  $\Gamma_p(0) = (-1)(-1) = 1$ . So  $\Gamma_p(0)\Gamma_p(1) = -1 = (-1)^1$  as desired. Again using part (a) we find that

$$\frac{\Gamma_p(s+1)\Gamma_p(1-(s+1))}{\Gamma_p(s)\Gamma_p(1-s)} = \frac{\Gamma_p(s+1)}{\Gamma_p(s)} \frac{\Gamma_p(-s)}{\Gamma_p(-s+1)}$$

is  $-1$  if  $s$  is not divisible by  $p$  and is  $1$  if  $s$  is divisible by  $p$ . The result follows for positive integers by induction, and for all of  $\mathbb{Z}_p$  by continuity.

- (c) For fixed  $m$ , let  $f(s)$  be the left-hand side of the desired identity and let  $g(s)$  be the right-hand side. Evidently  $f(1) = g(1)$ . Now

$$\frac{f(s+1)}{f(s)} = \frac{\Gamma_p(s)\Gamma_p((s/m)+1)}{\Gamma_p(s+1)\Gamma_p(s/m)}$$

is  $1/m$  if  $p$  does not divide  $s$ , and is  $1$  if  $p$  divides  $s$ . One verifies that the same is true of  $g(s+1)/g(s)$ , and the result follows.

5. Recalling that  $\mu_z((a+bp^N) + p^{N+1}\mathbb{Z}_p) = \frac{z^a + bp^N}{1-zp^{N+1}}$ , the result is an immediate consequence of the formula for summing (finitely many terms of a) geometric series.