Behavior of space periodic laminar flames near the extinction point

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Abstract

In this paper we study a free boundary problem, arising from a model for the propagation of laminar flames. Consider the heat equation

$$\Delta u = u_t, \;\; u > 0$$

in an unknown domain $\Omega \subset I\!R^n \times (0,T)$ for some $T > 0$ with the following boundary conditions

$$u = 0, \quad |\nabla u| = 1$$

on the lateral free boundary $\partial\Omega$. If the initial data u_0 is compactly supported, then the solution vanishes in a finite time T, which is called the extinction time. In this paper, we investigate the asymptotic behavior of a solution near the extinction time when the initial data u_0 is periodic in angle and the initial free boundary Γ_0 is contained in some annulus $B_M(0) \setminus B_1(0)$. Assuming small periodicity, we prove that the free boundary is asymptotically spherical with a quantitative estimate.

1 Introduction

In this paper we study a parabolic free boundary problem, which describes the propagation of equidiffusional premixed flames with high activation energy. The classical formulation is as follows. Let u_0 be a continuous and nonnegative initial function defined in $I\!R^n$, whose positive set is open and nonempty. We find a nonnegative continuous function u in $I\!R^n \times (0,T)$ such that

$$(P) \qquad \begin{cases} u_t = \Delta u & \text{in} \quad \{u > 0\} \\ |\nabla u| = 1, \; u = 0 & \text{on} \quad \partial\{u > 0\} \\ u(x,0) = u_0(x) \end{cases}$$

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where ∇u denotes the spatial gradient of u and $\{u > 0\}$ denotes the inverse image $\{(x,t) : u(x,t) > 0\}$. In combustion theory for laminar flames, u denotes the *minus temperature* $\lambda(T_c - T)$ where T_c is the flame temperature and λ is a normalization factor (see [BL]). The region $\Omega := \{u > 0\}$ represents the *unburnt zone* and the lateral free boundary Γ of Ω represents the *flame front.*

Assuming u_0 is bounded and Lipschitz continuous, a global weak solution of (P) has been obtained in [CV] as the asymptotic limit of the following approximation problems

$$(P_\epsilon) \qquad \begin{cases} \partial_t u_\epsilon = \Delta u_\epsilon + \dfrac{1}{\epsilon}\beta(\dfrac{u_\epsilon}{\epsilon}) \\ \\ u_\epsilon(x,0) = u_{0\epsilon}(x) \end{cases}$$

where β is a nonnegative smooth function supported on $[0,1]$ with $\int_0^1 \beta = 1/2$ and $u_{0\epsilon}$ approximates u_0 in a proper way. The family of solutions $\{u_\epsilon\}$ is uniformly bounded in $C^{1,1/2}_{x,t}$-norm on compact sets and they converges along subsequences to a function $u \in C^{1,1/2}_{loc}$, which is called a *limit solution* of (P). In this paper, we adopt the notion of a limit solution. Then it was proved by Kim [K] that the limit solution is unique and coincides with a *viscosity solution* if u has a shrinking support, i.e., if u_0 is a C^2-function with

$$\Delta u_0 \le 0 \text{ in } \Omega_0 \quad \text{and} \quad 0 \le |\nabla u_0| \le 1 \text{ on } \partial\Omega_0. \tag{1.1}$$

On the other hand, nonzero equilibria of (P) do not exist for compactly supported initial data and a solution with a bounded initial domain vanishes in a finite time T, i.e.,

$$u(x,T) \equiv 0, \quad u(x,t) \neq 0 \text{ for } 0 < t < T.$$

Here T is called the *extinction time*, the time when the unburnt zone collapses in a combustion model. Particularly, it was proved in [GHV] that if the initial data is radially symmetric and supported in a ball, the solution is asymptotically *self-similar* near the extinction time T. Also when $n = 1$, the profiles of any non-radial solution are asymptotically self-similar even if it might focus at a point other than 0. However in higher dimensions, not much is known for the behavior of non-radial solutions (see [BHS] and [BHL] for a linearized stability analysis). In general setting, it is expected that topological changes of the domain might occur, possibly generating multiple radial profiles for later times.

In this paper, we will investigate the asymptotic behavior of non-radial space-periodic solutions in higher dimensions $n \ge 2$. We will prove that, under appropriate assumptions on u_0 (see Remarks 2 and 3), the free boundary is asymptotically spherical near $(0,T) \in I\!R^{n+1}$. (Note that a solution focuses at the origin if it has a space periodic initial data supported on a simply connected set.) Then it will turn out that the solution is asymptotically self-similar and the free boundary is a graph of $C^{1+\gamma,\gamma}$ function after some positive time. For the existence, uniqueness of a solution (see [K]), we consider a natural situation (1.1) for the application in which u has a shrinking support at $t = 0$, i.e., we assume that the flame advances at the initial time. Below we state the main theorem of the paper, where $\Gamma_t(u) := \partial\{x : u(x,t) > 0\}$ denotes the free boundary of u at time t.

Theorem 1.1. *Let u be a solution of (P) with the initial data u_0 satisfying (1.1). Suppose*

$$u_0 = \phi_0 + \rho$$

where ϕ_0 is a nonnegative radial function supported on $B_1(0)$ and ρ is a nonnegative function supported on a subset of $B_M(0)$ for some $M > 1$. Suppose $\{\phi_0 + \rho > 0\}$ is simply connected, i.e., the initial free boundary $\Gamma_0(u)$ is a connected set contained in the annulus $B_M(0) \setminus B_1(0)$. Take $M > 0$ sufficiently large so that

$$|\nabla u_0| \leq M, \quad \max \phi_0 \geq 1/M.$$

Then there is a constant $\alpha(n,M) > 0$ depending only on M and dimension n such that if $\|\rho\|_\infty \leq \alpha$ and ρ is periodic in angle with period $\leq \alpha$ for some $0 < \alpha \leq \alpha(n,M)$, then the free boundary of u is asymptotically spherical near its focusing point $0 \in I\!R^n$.

More precisely, there exist constants $0 < h < 1$ and $C > 0$ depending on n and M such that for $t \in [(1-2^{-k})T, (1-2^{-k-1})T]$ and $k \geq 2$,

$$\Gamma_t(u) \subset B_{(1+Ch^k\alpha)r(t)}(0) - B_{r(t)}(0)$$

where $r(t)$ is a decreasing function of t with $r(T) = 0$.

Remark 1. The main theorem is different from a standard nonlinear stability analysis since

a. The initial free boundary $\Gamma_0(u) := \partial\{u_0 > 0\}$ is not assumed to be a slight perturbation of a sphere. It can be any irregular subset of the annulus $B_M(0) \setminus B_1(0)$, which is periodic in angle (not necessarily star-shaped).

b. Even if $\|\rho\|_\infty$ is assumed to be small, the function ρ can change the geometry of the free boundary $\Gamma_t(u)$ in a significant way near the initial time, since we do not assume any lower bound on $|\nabla\phi_0|$. Even when we start from a radially symmetric initial boundary $\Gamma_0(u) = \partial B_1(0)$, a small function ρ can change the geometry of $\Gamma_t(u)$ so that

$$\frac{\sup\{r : \partial B_r(0) \cap \Gamma_t(u) \neq \emptyset\}}{\inf\{r : \partial B_r(0) \cap \Gamma_t(u) \neq \emptyset\}}$$

is much larger than 1 for small $t > 0$. In fact, we prove that $\Gamma_t(u)$ is located between two concentric spheres with "just" comparable radii for $0 < t < T/2$ (Lemma 3.1). However for later times, $\Gamma_t(u)$ will be shown to get closer and closer to decreasing spheres $\partial B_{r(t)}(0)$ as t approaches the extinction time T (Proposition 8.1). (Here $r(T) = 0$.)

c. Even a small ρ can change the topology of the domain creating small pieces of the positive set around the main piece of the domain. However the topological change of the domain will not affect the geometry of the main piece eventually if ρ is assumed to be small.

d. There is no stability result on the extinction time T. A solution u focuses at a point with a divergent boundary speed, i.e., the speed of $\Gamma_t(u) \approx 1/\sqrt{T-t} \to \infty$ as $t \to T$.

Remark 2. Assuming an L^∞-bound on u_0, it was proved in [CV] that

$$|\nabla u(x,t)| \leq M$$

for $t > T/2$ and M depending on n and $\|u_0\|_\infty$. Hence, we suppose from the beginning that u_0 has bounded interior gradient, i.e., $|\nabla u_0| \leq M$. Also for simplicity, we assume that $\{u_0 > 0\}$ is simply connected. This assumption is used only in the proof of Lemma 3.1, and we expect that Lemma 3.1 will hold without the assumption.

Remark 3. If $\|\rho\|_\infty$ is not sufficiently small, we may have a dramatic change of the topology of the domain: splitting of the domain into several "large" pieces. Consider an example with an initial domain $B_1(0) \subset I\!R^2$, that large initial data around $(\pm 1/2, 0)$ leads to the split of the domain into two large pieces at a later time. Then for later times, we might get multiple profiles of the solution with similar sizes, focusing at different points. Also if ρ is not assumed to be periodic, the center of the the domain may change in time. For these technical reasons, we assume that ρ is periodic in angle with a small sup norm.

Central difficulty in analysis lies in rather difficult construction of various barrier functions near the focusing point. Here is an outline of the paper. In section 2 some preliminary lemmas and notations are introduced. In section 3 we show that the free boundary $\Gamma_t(u)$ is located between two concentric spheres with comparable radii at each time t, and prove that the maximal radial subregion Ω^{in} of Ω has a boundary close to a Lipschitz graph in a parabolic scaling. Then in section 4 the scaled α-flatness of the free boundary is obtained when the function ρ has size α. In other words, $\Gamma_t(u)$ is located between two concentric spheres with the outer radius bounded by $(1 + C\alpha)\times$ inner radius at later times t. In section 5 the solution u is approximated by a radial function at interior points away from the boundary, and this interior estimate is improved in section 6 thanks to the α-flatness of the free boundary. Then in section 7, the interior improvement (obtained in section 6) propagates to the free boundary at later times giving an improved estimate on the location of the free boundary. More precisely, if the free boundary is located near a sphere at each time $t \in ((1 - 2^{-k})T, T)$ then several barrier functions will show that the solution u gets closer to a radial function ϕ at later times $t \in ((1 - 2^{-k-1})T, T)$, at points away from the boundary (Lemma 6.1). This improved estimate on the values of u forces the free boundary to be located in a smaller neighborhood of a sphere at $t \in ((1 - 2^{-k-2})T, T)$ (Lemma 7.1). By iteration, it will follow that the free boundary is asymptotically spherical near the focusing point. In the last section, the asymptotic behavior of the solution is investigated and the regularity of the free boundary follows as a corollary from the flatness of the free boundary and the radial approximations of the solution.

2 Preliminary lemmas and notations

Below we introduce some notations.

• Denote by $\Omega(u)$, the positive set of u, i.e.,

$$\Omega(u) = \{u > 0\} = \{(x, t) : u(x, t) > 0\}.$$

• Denote by $\Gamma(u)$, the free boundary of u, i.e.,

$$\Gamma(u) = \partial\Omega(u) \cap \{t > 0\}.$$

• Denote by Σ_t, the time cross section of a space time region Σ, i.e.,

$$\Sigma_t = \{x : (x, t) \in \Sigma\}.$$

In particular,

$$\Omega_t(u) = \{x : u(x, t) > 0\}$$

and

$$\Gamma_t(u) = \partial\Omega_t(u)$$

where $\Gamma_t(u)$ is called the free boundary of u at time t.

• Denote by $B_r(x)$, the space ball with radius r, centered at x.

• Denote by $Q_r(x,t)$, the parabolic cube with radius r, centered at (x,t). Denote by $Q_r^-(x,t)$, its negative part, i.e.,

$$Q_r(x,t) = B_r(x) \times (t-r^2, t+r^2), \quad Q_r^-(x,t) = B_r(x) \times (t-r^2, t).$$

• Denote by $K_r(x,t)$, the hyperbolic cube with radius r, centered at (x,t), i.e.,

$$K_r(x,t) = B_r(x) \times (t-r, t+r).$$

• A space time region Ω is Lipschitz in $Q_r(0)$ (in parabolic scaling) if

$$Q_r(0) \cap \Omega = Q_r(0) \cap \{(x,t) : x_n > f(x',t)\}$$

where $x = (x', x_n) \in I\!R^{n-1} \times I\!R$ and f satisfies

$$|f(x',t) - f(y',s)| \leq L(|x'-y'| + |t-s|/r)$$

for some $L > 0$, with $f(0,0) = 0$.

• A function $\rho : I\!R^n \to I\!R$ is periodic in angle with period $\leq \alpha$ if

$$\rho(r, \theta_1, ..., \theta_i + p_i, ..., \theta_{n-1}) = \rho(r, \theta_1, ..., \theta_i, ..., \theta_{n-1})$$

for $0 \leq p_i \leq \alpha$ and $1 \leq i \leq n-1$.

• Denote by $r^{in}(t)$, the maximal radius of a circle centered at the origin which is inscribed in $\Omega_t(u)$, i.e.,

$$r^{in}(t) = \sup\{r : B_r(0) \subset \Omega_t(u)\}.$$

• Denote by $r^{out}(t)$, the minimal radius of a circle centered at the origin, in which $\Omega_t(u)$ is inscribed, i.e.,

$$r^{out}(t) = \inf\{R : \Omega_t(u) \subset B_R(0)\}.$$

• Denote by Ω^{in}, the maximal radial region inscribed in $\Omega(u)$, i.e., its time cross section Ω_t^{in} is given by

$$\Omega_t^{in} = B_{r^{in}(t)}(0) \text{ for all } t.$$

• Denote by $0 < t_1 < t_2 < ... < T$, the diadic decomposition of the time interval $(0, T)$ such that $t_i = (1 - 2^{-i})T$, i.e.,

$$t_1 = T/2\ ,\ \ t_{i+1} - t_i = \frac{T - t_i}{2}.$$

• For positive numbers a and b, write $a \approx b$ if there exist positive constants C_1 and C_2 depending only on n and M such that

$$C_1 a \le b \le C_2 a.$$

Below we state some properties of caloric functions defined in Lipschitz domains, a comparison principle, results on existence of self-similar solutions and asymptotic behavior of radial solutions, and a regularity result for solutions with flat boundaries.

Lemma 2.1. *[ACS, Lemma 5] Let Ω be a Lipschitz domain in $Q_1(0)$ such that $0 \in \partial\Omega$. Let u be a positive caloric function in $Q_1(0) \cap \Omega$ such that $u = 0$ on $\partial\Omega$, $u(e_n, 0) = m_1 > 0$ and $\sup_{Q_1(0)} u = m_2$. Then there exist $a > 0$ and $\delta > 0$ depending only on n, L, m_1/m_2 such that*

$$w_+ := u + u^{1+a} \text{ and } w_- := u - u^{1+a}$$

are, respectively, subharmonic and superharmonic in $Q_\delta \cap \Omega \cap \{t = 0\}$.

Lemma 2.2. *[ACS, Theorem 2] Let Ω and u be given as in Lemma 2.1, then for every $\mu \in \{\mu \in I\!\!R^{n+1} : |\mu| = 1, e_n \cdot \mu < \cos\theta\}$ where $\theta = \cot^{-1}(L)/2$, $D_\mu u > 0$ in $Q_\delta \cap \Omega$ for a positive constant δ depending on n, L, m_1/m_2 and $\|\nabla u\|_{L^2}$.*

Lemma 2.3. *[ACS, Corollary 4] Let Ω and u be given as in Lemma 2.1, then there exist positive constants c_1 and c_2 depending on n and L such that*

$$c_1 \frac{u(x,t)}{d_{x,t}} \le |(\nabla_x, \partial_t) u| \le c_2 \frac{u(x,t)}{d_{x,t}}$$

for every $(x, t) \in K_r(0) \cap \Omega$, where $d_{x,t}$ is the elliptic distance from (x, t) to $\partial\Omega$.

Lemma 2.4. *[D, Theorem 12.2] Let u be a caloric function in $Q_\delta = Q_\delta(0)$, then there exists a dimensional constant $C > 0$ such that*

$$\|\nabla u\|_{\infty, Q^-_{\sigma\delta}} \le \frac{C}{(1-\sigma)^{n+3}\delta|Q^-_\delta|} \int_{Q^-_\delta} |u| dxdt,$$

$$\|u_t\|_{\infty, Q^-_{\sigma\delta}} \le \frac{C}{(1-\sigma)^{n+4}\delta^2|Q^-_\delta|} \int_{Q^-_\delta} |u| dxdt$$

for $\sigma \in (0, 1)$ where $|Q^-_\delta|$ is the volume of Q^-_δ.

Lemma 2.5. *[CV, Theorem 4.1] Let u be a limit solution of (P), for which the initial function u_0 is nonnegative, bounded and $|\nabla u_0| \leq M$. Then for a dimensional constant $C_0 > 0$*

$$|\nabla u| \leq C_0 \max\{1, M\}.$$

Lemma 2.6. *[K, Theorem 1.3 and Theorem 2.2] Let u an v be, respectively, a sub- and supersolutions of (P) with strictly separated initial data $u_0 \prec v_0$. Then the solution remain ordered for all time, i.e.,*

$$u(x,t) \prec v(x,t) \text{ for every } t > 0.$$

Lemma 2.7. *[CV, Proposition 1.1] Let $T > 0$. Then there exists a self similar solution $U(x,t)$ of (P) in the form*

$$U(x,t) = (T-t)^{1/2} f(|x|/(T-t)^{1/2})$$

where the profile $f(r)$ satisfies the stationary problem

$$f'' + (\frac{n-1}{r} - \frac{1}{2}r)f' + \frac{1}{2}f = 0 \text{ for } 0 < r < R,$$

$$f'(0) = 0 \text{ and } f(r) > 0 \text{ for } 0 \leq r < R$$

with boundary conditions

$$f(R) = 0 \text{ and } f'(R) = -1.$$

Lemma 2.8. *[GHV, Theorem 6.6] Let u be a radial solution of (P) with initial data $u_0 = u_0(|x|) > 0$ supported in a ball. Then*

$$(T-t)^{-1/2} u(|x|, t) \to f(|x|/(T-t)^{1/2}) \text{ uniformly}$$

as $t \to T$ with f given as in Lemma 2.7.

Lemma 2.9. *[AW, Theorem 8.4] Let (u, χ) be a domain variation solution of (P) in $Q_\rho := Q_\rho(0,0)$ such that $(0,0) \in \partial\{u > 0\}$. There exists a constant $\sigma_1 > 0$ such that if $u(x,t) = \chi(x,t) = 0$ when $(x,t) \in Q_\rho^-$ and $x_n \geq \sigma\rho$, and if $|\nabla u| \leq 1 + \tau$ in Q_ρ^- for some $\sigma \leq \sigma_1$ and $\tau \leq \sigma_1 \sigma^2$, then the free boundary $\partial\{u > 0\}$ is in $Q_{\rho/4}^-$ the graph of a $C^{1+\gamma,\gamma}$-function; in particular the space normal is Hölder continuous in $Q_{\rho/4}^-$.*

3 Estimate on inner and outer radii of Ω

If U is a self-similar solution (see Lemma 2.7) with an extinction time T, then the maximum of U at time t and the radius of its support $\Omega_t(U)$ are constant multiples of $\sqrt{T-t}$. In this section we prove analogous estimates on $\max u(\cdot,t)$ and on inner, outer radii of concentric spheres which trap the free boundary $\Gamma_t(u)$ in between. Recall that $r^{in}(t)$ and $r^{out}(t)$ are inner and outer radii of $\Omega_t(u)$, i.e.,

$$r^{in}(t) = \sup\{r : B_r(0) \subset \Omega_t(u)\}$$

and

$$r^{out}(t) = \inf\{R : \Omega_t(u) \subset B_R(0)\}.$$

Lemma 3.1. *Let u_0 be as in Theorem 1.1 and let $\alpha \le \alpha(n,M)$ for a sufficiently small $\alpha(n,M) > 0$. Then there exist constants C_j ($1 \le j \le 4$) depending on n and M such that*

$$C_1\sqrt{T-t} \le \max u(\cdot,t) \le C_2\sqrt{T-t} \tag{3.1}$$

and

$$C_3\sqrt{T-t} \le r^{in}(t) \le r^{out}(t) \le C_4\sqrt{T-t}. \tag{3.2}$$

Proof. We prove the last inequality of (3.1) and the first inequality of (3.2) by comparison with self similar solutions. Then using these inequalities, we prove the first and the last inequality of (3.1) and (3.2). Without loss of generality, assume $M \ge 1$.

1. *Proof of* $\max u(\cdot,t) \le C_2\sqrt{T-t}$ *and* $C_3\sqrt{T-t} \le r^{in}(t)$: Let T be the extinction time of u. By Lemma 2.7, there exists a self-similar solution v with the extinction time T. Then for positive dimensional constants a_1 and a_2,

$$\max v(\cdot,t) = a_1\sqrt{T-t} \quad\text{and}\quad \Omega_t(v) = B_{a_2\sqrt{T-t}}(0).$$

To find an upper bound on $u(\cdot,t)$, suppose that for some $x_0 \in \Omega_t(u)$

$$u(x_0,t) \ge (2a_1 + C_0Ma_2)\sqrt{T-t}$$

where C_0 is the constant as in Lemma 2.5. Then by Lemma 2.5

$$u(\cdot,t) \ge 2a_1\sqrt{T-t} = 2\max v(\cdot,t)$$

on $B_{a_2\sqrt{T-t}}(x_0) = \Omega_t(v(x-x_0,t))$. By comparing u with $v(x-x_0,t)$,

$$\max u(x,T) > \max v(x-x_0,T) = 0$$

which would contradict that u vanishes at time T. Hence we obtain

$$\max u(\cdot,t) \le C_2\sqrt{T-t} \tag{3.3}$$

with $C_2 = 2a_1 + C_0Ma_2$. The first inequality of (3.2), that is $C_3\sqrt{T-t} \le r^{in}(t)$, can be proved similarly for $C_3 = C_3(n,M) > 0$ by comparing u with a self-similar solution.

2. *Proof of* $C_1\sqrt{T-t} \le \max u(\cdot,t)$ *and* $r^{out}(t) \le C_4\sqrt{T-t}$: These inequalities will be proved simultaneously by induction. Let C_4 be a sufficiently large constant depending on M and n, which will be determined later. Recall that $0 < t_1 < t_2 < \ldots < T$ is a diadic decomposition of $(0,T)$ with $t_1 = T/2$ and $t_{i+1} - t_i = (T - t_i)/2$.

Claim 1. Suppose

$$r^{out}(t_i) \le C_4\sqrt{T-t_i} \tag{3.4}$$

for some $i \in \mathbb{N}$, then

$$\max u(\cdot,t_i) \ge \frac{C_nC_3^2}{C_4}\sqrt{T-t_i}$$

where C_n is a positive dimensional constant and C_3 is the constant as in the first inequality of (3.2). For the proof of Claim 1, we construct a Lipschitz region Σ in $I\!R^n \times [t_i, t_{i+1}]$. Since $r^{out}(t)$ is decreasing in time t, there exists a decreasing function $\sigma(t)$ on $[t_i, t_{i+1}]$ such that

(a-1) $\sigma(t) \ge r^{out}(t)$

(a-2) $\sigma(\tau) = r^{out}(\tau)$ for some $\tau \in [(t_i + t_{i+1})/2, t_{i+1}]$

(a-3) $|\sigma'(t)| \le \dfrac{2(r^{out}(t_i) - r^{out}(t_{i+1}))}{t_{i+1} - t_i}$.

(We can construct $\sigma(t)$ so that it is linear on $[(t_i + t_{i+1})/2, t_{i+1}]$ with slope $-2(r^{out}(t_i) - r^{out}(t_{i+1}))/(t_{i+1} - t_i)$ and it is a constant on $[t_i, (t_i + t_{i+1})/2]$.) Let Σ be a space-time region in $I\!R^n \times [t_i, t_{i+1}]$ such that its time cross section is a ball of radius $\sigma(t)$ centered at 0, i.e.,

$$\Sigma_t = B_{\sigma(t)}(0)$$

for $t_i \le t \le t_{i+1}$. Then the properties (a-1), (a-2) and (a-3) imply

(b-1) $\Omega(u) \cap \{t_i \le t \le t_{i+1}\} \subset \Sigma$

(b-2) There exists a free boundary point

$$p \in \partial B_{\sigma(\tau)}(0) \cap \Gamma_\tau(u),$$

i.e., $(p, \tau) \in \partial\Sigma \cap \Gamma(u)$.

(b-3) Σ is Lipschitz in space and time with a Lipschitz constant

$$L := \frac{2\sigma(t_i)}{t_{i+1} - t_i}$$

where (b-2) follows from (a-2) since $\partial B_{r^{out}(t)}(0)$ intersects $\Gamma_t(u)$ for all t.

Define a function $w_{t_i}(x)$ on $B_{\sigma(t_i)}(0)$ by

$$w_{t_i}(x) = \begin{cases} \max u(\cdot, t_i) & \text{for} \quad x \in B_{\sigma_0}(0) \\ \max u(\cdot, t_i) - C_0 M|x| & \text{for} \quad x \in B_{\sigma(t_i)}(0) - B_{\sigma_0}(0) \end{cases}$$

where C_0 is the constant as in Lemma 2.5 and σ_0 is chosen so that $w_{t_i} = 0$ on $\partial B_{\sigma(t_i)}(0)$. Then by Lemma 2.5 and by $\Omega_{t_i}(u) \subset B_{r^{out}(t_i)}(0) \subset B_{\sigma(t_i)}(0)$,

$$u(\cdot, t_i) \leq w_{t_i}(\cdot).$$

Let $w(x,t)$ be a caloric function in Σ such that

$$\begin{cases} \Delta w = w_t & \text{in} \quad \Sigma \\ w = w_{t_i} & \text{on} \quad \{t = t_i\} \\ w = 0 & \text{on} \quad \partial\Sigma \cap \{t_i < t < t_{i+1}\}. \end{cases}$$

Then by comparison, $w \geq u$ in Σ. Since $w(p,\tau) = u(p,\tau) = 0$, the inequality $w \geq u$ implies

$$|\nabla w(p, \tau)| \geq 1. \tag{3.5}$$

Denote

$$\sigma(t_i) = \beta\sqrt{t_{i+1} - t_i}$$

for some $\beta > 0$. Then

$$L = \frac{2\beta}{\sqrt{t_{i+1} - t_i}}. \tag{3.6}$$

Also observe that $r^{out}(t_i) \leq \sigma(t_i) \leq 2r^{out}(t_i)$ by the construction of $\sigma(t)$. This implies

$$C_3 < \beta < 4C_4 \tag{3.7}$$

where the first inequality follows from the first inequality of (3.2) and the last inequality follows from the assumption (3.4). Here we assume that $C_3 \le 1$ without loss of generality.

Since $\partial\Sigma = \partial\{w(x,t) > 0\}$ has a Lipschitz constant L, the caloric function

$$\tilde{w}(x,t) := w(\frac{x}{L}, \tau + \frac{t}{L^2})$$

has a Lipschitz boundary with Lipschitz constant 1 in the region

$$B_{C_3\beta}(Lp) \times [-\beta^2, 0].$$

Since $\beta > C_3$ and $C_3 \le 1$, $\tilde{w}$ has a Lipschitz boundary in a smaller region

$$Q^-_{C_3^2}(Lp, 0) := B_{C_3^2}(Lp) \times [-C_3^4, 0]$$

with a Lipschitz constant 1. Hence by Lemma 2.1, $\tilde{w}(\cdot, 0)$ is almost harmonic near the vanishing Lipschitz boundary $\partial B_{Lr^{out}(\tau)}(0)$. More precisely, there exists a constant $0 < C_n < 1$ depending on n such that the following holds: if h is a harmonic function in the annulus $B_{Lr^{out}(\tau)}(0) - B_{Lr^{out}(\tau) - C_n C_3^2}(0)$ with

$$h = \begin{cases} 0 & \text{on} \quad \partial B_{Lr^{out}(\tau)}(0) \\ \\ 2\tilde{w}(\cdot, 0) & \text{on} \quad \partial B_{Lr^{out}(\tau) - C_n C_3^2}(0) \end{cases}$$

then on $B_{Lr^{out}(\tau)}(0) - B_{Lr^{out}(\tau) - C_n C_3^2}(0)$

$$\tilde{w}(\cdot, 0) \le h(\cdot). \tag{3.8}$$

Combining (3.5) and (3.8), we obtain

$$|\nabla h(Lp)| \ge |\nabla \tilde{w}(Lp, 0)| = |\nabla w(p, \tau)|/L \ge 1/L. \tag{3.9}$$

This implies

$$\begin{aligned} \max u(\cdot, t_i) &= \max w(\cdot, t_i) \\ &\ge \max w(\cdot, \tau) \\ &= \max \tilde{w}(\cdot, 0) \\ &\ge \max\{\tilde{w}(x, 0) : x \in \partial B_{Lr^{out}(\tau) - C_n C_3^2}(0)\} \\ &= \max h/2 \\ &\ge C_n C_3^2 / L \\ &\ge \frac{C_n C_3^2}{C_4} \sqrt{T - t_i} \end{aligned} \tag{3.10}$$

where the third inequality follows from (3.9) for a dimensional constant C_n and the last inequality follows from (3.6), (3.7) and $T - t_i = 2(t_{i+1} - t_i)$.

Claim 2. Suppose $r^{in}(t_i) \le (C_4/8)\sqrt{T - t_i}$ for $1 \le i \le k$, then

$$r^{out}(t_i) \le C_4\sqrt{T - t_i} \tag{3.11}$$

for $1 \le i \le k$. We prove Claim 2 by induction. Let x_0 be a point in $\Omega_0(u)$ such that $\max u_0 = u_0(x_0)$. Since $\max u_0 \ge 1/M$ and $|\nabla u_0| \le M$,

$$u_0 \ge \frac{1}{2M} \quad \text{on} \quad B_{1/2M^2}(x_0).$$

Then by comparing u with a self similar solution,

$$T \ge C(n, M) \tag{3.12}$$

where $C(n, M)$ is a constant depending on n and M. Hence

$$r^{out}(t_1) \le r^{out}(0) \le M \le C_4\sqrt{T - t_1}$$

where the first inequality follows since $\Gamma_t(u)$ shrinks in time, the second inequality follows from the assumption on u_0 and the last inequality follows from (3.12) if $C_4 = C_4(n, M)$ is chosen sufficiently large.

Now suppose that (3.11) holds for $i \in \{1, ..., j\}$ where $j \le k - 1$. Construct a Lipschitz region Σ in $\mathbb{R}^n \times [t_j, t_{j+1}]$ as in the proof of Claim 1 so that

$$\Omega(u) \cap \{t_j \le t \le t_{j+1}\} \subset \Sigma, \quad (p, \tau) \in \Gamma(u) \cap \partial\Sigma.$$

If $r^{out}(t_{j+1}) > C_4\sqrt{T - t_{j+1}}$, then

$$r^{out}(\tau) \ge r^{out}(t_{j+1}) > \frac{C_4}{2}\sqrt{T - \tau} > \frac{C_4}{4}\sqrt{T - t_j} \ge 2r^{in}(t_j) \tag{3.13}$$

where the last inequality follows from the assumption on $r^{in}(t_j)$. Let

$$\tilde{\Sigma} := \Sigma - \Omega^{in}$$

where Ω^{in} is the region constructed in Section 2, i.e., Ω^{in} is the maximal radial region inscribed in $\Omega(u)$. Then by (3.13) and (b-3), $\tilde{\Sigma}$ is Lipschitz in the "large" cube $Q^-_{C_4\sqrt{T-\tau}/4}(p, \tau)$. Let $v(x, t)$ solve

$$\begin{cases} \Delta v = v_t & \text{in} \quad \tilde{\Sigma} \\ v = \max_{\tilde{\Sigma}_{t_j}} u(\cdot, t_j) & \text{on} \quad \{t = t_j\} \\ v = 0 & \text{on} \quad \partial\Sigma \cap \{t_j < t < t_{j+1}\} \\ v(\cdot, t) = \max_{\partial B_{r^{in}(t)}(0)} u(\cdot, t) & \text{on} \quad \partial\Omega^{in} \cap \{t_j < t < t_{j+1}\}. \end{cases}$$

Then by comparison, $v \geq u$ in $\tilde{\Sigma}$. By a similar argument as in the proof of (3.10) with w replaced by v, and with (3.4) replaced by the assumption $r^{out}(t_j) \leq C_4\sqrt{T - t_j}$, we obtain

$$\max v(\cdot, \tau) \geq \frac{C_n C_3^2}{C_4}\sqrt{T - t_j}. \tag{3.14}$$

On the other hand, observe that for a dimensional constant C_n

$$\max_{\partial B_{r^{in}(t)}(0)} u(\cdot, t) \leq C_n M \alpha r^{in}(t) \tag{3.15}$$

since $\min_{\partial B_{r^{in}(t)}(0)} u(\cdot, t) = 0$, u is periodic in angle with period $< \alpha$ and $|\nabla u| \leq C_0 M$ (Lemma 2.5). Similarly,

$$\max_{\tilde{\Sigma}_{t_j}} u(\cdot, t_j) = \max_{\substack{\partial B_s(0) \\ r^{in}(t_j) \leq s \leq r^{out}(t_j)}} u(\cdot, t_j) \leq C_n M \alpha r^{out}(t_j) \tag{3.16}$$

since the simple connectivity of $\Omega_0(u)$ and $u_t \leq 0$ imply that $\min_{\partial B_s(0)} u(\cdot, t_j) = 0$ for $r^{in}(t_j) \leq s \leq r^{out}(t_j)$. Hence

$$\max v \leq C_n M \alpha r^{out}(t_j) \leq C_n M \alpha C_4 \sqrt{T - t_j}$$

where the last inequality follows from the assumption on $r^{out}(t_j)$. If $\alpha(n, M)$ is chosen sufficiently small, then the above upper bound on $\max v$ would contradict (3.14). Hence we conclude

$$r^{out}(t_{j+1}) \leq C_4\sqrt{T - t_{j+1}}.$$

Claim 3. If C_4 is chosen sufficiently large,

$$r^{in}(t_i) \leq \frac{C_4}{8}\sqrt{T - t_i} \tag{3.17}$$

for all $i \geq 1$. For $i = 1$, (3.17) follows from (3.12) and $r^{in}(t_1) \leq r^{in}(0) = 1$. Now suppose that (3.17) holds for $1 \leq i \leq j$ and not for $i = j + 1$, i.e.,

$$r^{in}(t_{j+1}) := r_0 > \frac{C_4}{8}\sqrt{T - t_{j+1}}. \tag{3.18}$$

Since (3.17) holds for $1 \leq i \leq j$, Claim 2 implies

$$r^{out}(t_i) \leq C_4\sqrt{T - t_i} \text{ for } 1 \leq i \leq j.$$

Then by Claim 1,

$$\max u(\cdot,t_i) \geq \frac{C_n C_3^2}{C_4}\sqrt{T-t_i} \text{ for } 1 \leq i \leq j.$$

Since (3.16) implies that $\max u(\cdot,t_i)$ $(1 \leq i \leq j)$ is taken inside the maximal radial region $\Omega^{in} \subset \Omega(u)$, and since $T \geq C(n,M)$,

$$u(0,t_i) \geq \frac{CC_3^2}{C_4}\sqrt{T-t_i} \text{ for } 1 \leq i \leq j \tag{3.19}$$

where C is a constant depending on n and M.

Let $k = \min\{k \in \{1,...,j\} : t_{j+1} - t_k \leq r_0^2\}$ where $r_0 = r^{in}(t_{j+1})$. (Here observe that $t_{j+1} - t_j = T - t_{j+1} < r_0^2$ by (3.18).) Then

$$B_{r_0}(0) \times [t_k, t_{j+1}] \subset Q^-_{r_0}(0,t_{j+1}) \subset \Omega(u). \tag{3.20}$$

Observe that

$$\begin{cases} t_{j+1} - t_k \geq (t_{j+1} - t_{k-1})/3 \geq r_0^2/3 & \text{if} \quad k \neq 1 \\ \\ t_{j+1} - t_k \geq T/4 \geq C(n,M) \geq C(n,M) r_0^2 & \text{if} \quad k = 1. \end{cases} \tag{3.21}$$

Then by (3.18), (3.19), (3.20) and (3.21)

$$\begin{aligned} \min_{B_{r_0/2}(0)} u(\cdot,t_{j+1}) &\geq Cu(0,t_k) \\ &\geq \frac{CC_3^2}{C_4}\sqrt{T-t_k} \\ &\geq \frac{CC_3^2}{C_4}\sqrt{t_{j+1}-t_k} \\ &\geq \frac{CC_3^2}{C_4}\cdot r_0 \\ &\geq CC_3^2\sqrt{T-t_{j+1}} \end{aligned}$$

where C denote constants depending on n and M. In other words, $u(\cdot,t_{j+1})$ has a lower bound $CC_3^2\sqrt{T-t_{j+1}}$ on the large ball $B_{r_0/2}(0)$ with the radius

$$r_0/2 \geq C_4\sqrt{T-t_{j+1}}/8.$$

Hence if $C_4 = C_4(n,M,C_3)$ is chosen sufficiently large, then u would have an extinction time larger than T, contradicting $u(\cdot,T) \equiv 0$. Hence we conclude $r^{out}(t_{j+1}) \leq (C_4/8)\sqrt{T-t_{j+1}}$.

□

If U is a self-similar solution with an extinction time T, then the normal velocity of its free boundary at time t is comparable to $1/\sqrt{T-t} \approx 1/r^{in}(t)$, and hence $\Gamma(U)$ is Lipschitz in a parabolic scaling in each $Q_{r^{in}(t)}(x,t)$, $(x,t) \in \Gamma(U)$. Recall that $\Omega^{in} = \Omega^{in}(u)$ is the maximal radial subregion of $\Omega(u)$, i.e., its time cross section Ω^{in}_t is given by

$$\Omega^{in}_t = B_{r^{in}(t)}(0) \text{ for } 0 \le t \le T.$$

In the next lemma, we prove an analogous result that the average normal velocity of $\partial\Omega^{in}$ is bounded above by $C(n,M)/\sqrt{T-t}$ on each time interval $[t, t+\alpha r^{in}(t)^2]$ for $t \ge T/2$. This gives that the inner region Ω^{in} can be approximated by a subregion Ω_1 which is Lipschitz in a parabolic scaling.

Lemma 3.2. *Let u_0 be as in Theorem 1.1 and let $\alpha \le \alpha(n,M)$ for a sufficiently small $\alpha(n,M) > 0$. Then there exists a space-time region $\Omega_1 \subset \Omega^{in}$, which is radial in space and satisfies the following conditions:*

(i) For $i \ge 1$ (i.e., for $t_i \ge T/2$)

$$S_i := \Omega_1 \cap \{t_i \le t \le t_{i+1}\}$$

is Lipschitz in a parabolic scaling with a Lipschitz constant $C(n,M)$, i.e., the normal velocity of ∂S_i is bounded above by $C(n,M)/r^{in}(t_i)$.

(ii) ∂S_i is located in the $C(n,M)\alpha r^{in}(t_i)$-neighborhood of $\partial\Omega^{in}$.

In particular,

$$u \le C(n,M)\alpha r^{in}(t_i) \text{ on } \partial S_i. \tag{3.22}$$

Proof. Suppose that we have a subregion Ω_1 of Ω^{in} satisfying the conditions (i) and (ii). Then for a constant C depending on n and M,

$$\max_{\partial S_i} u \le \max_{\partial\Omega^{in} \cap \{t_i \le t \le t_{i+1}\}} u + C\alpha r^{in}(t_i) \le C\alpha r^{in}(t_i)$$

where the first inequality follows from $|\nabla u| \le C_0 M$ (Lemma 2.5), and the last inequality follows from (3.15).

Denote by V_I, the average normal velocity of $\partial\Omega^{in}$ on the time interval I. Decompose the time interval $[t_i, t_{i+1}]$ into subintervals of length $\alpha r^{in}(t_i)^2$, i.e., let

$$t_i = t_{i0} < t_{i1} = t_{i0} + \alpha r^{in}(t_i)^2 < t_{i2} = t_{i1} + \alpha r^{in}(t_i)^2 < ... < t_{ik} = t_{i+1}.$$

For the construction of Ω_1 satisfying the condition of the lemma, it suffices to prove that

$$V_{[t_{ij},\, t_{i\,j+1}]} \leq C(n,M)/r^{in}(t_i). \tag{3.23}$$

More precisely, given the estimate (3.23), the Lipschitz subregion Ω_1 can be constructed so that $\partial\Omega_1$ and $\partial\Omega^{in}$ intersect at times $t = s_m$ with $\{s_m\} \subset [t_i, t_{i+1}]$ and $|s_{m+1} - s_m| \leq \alpha r^{in}(t_i)^2$.

Below we prove (3.23). Let $i \geq 1$. By Lemma 3.1, the average velocity of $\partial\Omega^{in}$ on $[t_{i-1}, t_i]$, that is $V_{[t_{i-1},t_i]}$, is bounded above by

$$C(n,M)/\sqrt{T - t_i} \approx C(n,M)/r^{in}(t_i).$$

Then there exists $\tau \in [(t_{i-1}+t_i)/2, t_i]$ such that $V_{[t,\tau]} \leq C(n,M)/r^{in}(t_i)$ for all $t \in [t_{i-1}, \tau]$. In particular, $V_{[\tau-\alpha r^{in}(t_i)^2,\tau]} \leq C(n,M)/r^{in}(t_i)$. Let

$$\Sigma = B_{r^{in}(\tau)}(0) \times [\tau - \alpha r^{in}(t_i)^2, \tau].$$

Denote $\tilde{\tau} = \tau - \alpha r^{in}(t_i)^2$ and let $\tilde{\phi}(x,t)$ be the maximal radial function such that $\tilde{\phi}(x,t) \leq u(x,t)$. Let $\psi(x,t)$ be a solution of

$$\begin{cases} \Delta\psi = \psi_t & \text{in} \quad \Sigma \\ \psi = 0 & \text{on} \quad \partial\Sigma \\ \psi = \tilde{\phi} & \text{on} \quad \{t = \tilde{\tau}\}. \end{cases}$$

Then $\psi \leq u$ and by Lemma 2.1, $\psi(\cdot,\tau)$ is almost harmonic in the $c\sqrt{\alpha} r^{in}(t_i)$-neighborhood of $\partial\Omega^{in}_\tau$. Observe that $\partial\Sigma$ is located in the $C\alpha r^{in}(t_i)$-neighborhood of $\partial\Omega^{in}$, since $V_{[\tilde{\tau},\tau]} \leq C(n,M)/r^{in}(t_i)$. Then by a similar argument as in (3.15) and (3.16),

$$u - \psi = u \leq C(n,M)\alpha r^{in}(t_i) \;\; \text{in} \;\; \Omega \cap \{\tilde{\tau} \leq t \leq \tau\} - \Sigma.$$

Also since the initial perturbation $\rho \leq \alpha \leq \alpha M \max\phi_0$,

$$\psi \geq (1 - C(n,M)\alpha)u \;\; \text{on} \;\; \Sigma_{\tilde{\tau}}.$$

Hence on $\partial B_{(1-c\sqrt{\alpha})r^{in}(t_i)}(0)$,

$$\psi(\cdot,\tau) \geq u(\cdot,\tau) - C(n,M)\alpha r^{in}(t_i) \geq (1 - C\sqrt{\alpha})u(\cdot,\tau) \tag{3.24}$$

and $|\nabla\psi| \geq 1 - C\sqrt{\alpha}$ on $\partial\Omega^{in}_\tau$ for a constant C depending on n and M. (Otherwise, there would exist a free boundary point at which $|\nabla u| < 1$.) Hence

$u(\cdot,\tau)$ is bounded below by a function $\psi(\cdot,\tau)$ which is almost harmonic in the $c\sqrt{\alpha}r^{in}(t_i)$-neighborhood of $\partial\Omega^{in}_\tau$ with $|\nabla\psi| \geq 1 - C\sqrt{\alpha}$ on $\partial\Omega^{in}_\tau$. In fact, this lower bound can be obtained with α replaced by any number $\geq \alpha$. Then by a barrier argument, $u(\cdot,\tau+\alpha r^{in}(t_i)^2) > 0$ on $B_{r^{in}(\tau)-C\alpha r^{in}(t_i)}(0)$, i.e.,

$$V_{[\tau,\tau+\alpha r^{in}(t_i)^2]} \leq C(n,M)/r^{in}(t_i).$$

Next, we take $\tau_1 \in [\tau+\alpha r^{in}(t_i)^2/2, \tau+\alpha r^{in}(t_i)^2]$ such that $V_{[t,\tau_1]} \leq C(n,M)/r^{in}(t_i)$ for all $t \in [\tau,\tau_1]$. Then by a similar argument, we obtain (3.23) on the interval $[\tau_1, \tau_1 + \alpha r^{in}(t_i)^2]$. By induction, (3.23) holds on each time interval with length $\alpha r^{in}(t_i)^2$. □

4 α-flatness of free boundary

In this section we prove the α-flatness of the free boundary $\Gamma(u)$. More precisely, for $(x,t) \in \Gamma(u)$ we locate the free boundary part $\Gamma(u)\cap Q^-_{r(t)}(x,t)$ in the $K\alpha r(t)$-neighborhood of the Lipschitz boundary $\partial\Omega_1$. Here α is the size of the initial perturbation and K is a constant depending on n and M.

- Denote by $r(t)$, the radius of the time cross-section of Ω_1 at time t, i.e.,

$$\Omega_{1t} = B_{r(t)}(0) \text{ for } 0 \leq t \leq T.$$

Note that by Lemma 3.2, $r(t) \leq r^{in}(t) \leq (1+C\alpha)r(t)$ for a constant C depending on n and M.

Lemma 4.1. *Let u_0 be as in Theorem 1.1 and let $\alpha \leq \alpha(n,M)$ for a sufficiently small $\alpha(n,M) > 0$. Then*

$$r(t) \leq r^{out}(t) \leq (1+K\alpha)r(t)$$

for $t_2 \leq t \leq T$, and for a constant $K > 0$ depending on n and M. In other words, $\Gamma_t(u)$ is contained in the annulus $B_{(1+K\alpha)r(t)}(0) - B_{r(t)}(0)$ for $t_2 \leq t \leq T$.

Proof. Let K be a sufficiently large constant depending on n and M, which will be chosen later. Let Ω_2 be a space time region containing Ω_1 such that its time cross-section Ω_{2t} is given by

$$\Omega_{2t} = (1+K\alpha)\Omega_{1t} := B_{(1+K\alpha)r(t)}(0)$$

for $0 \leq t \leq T$. Since Ω_1 is Lipschitz in a parabolic scaling for $t \geq t_1$, Ω_2 is also Lipschitz in a parabolic scaling for $t \geq t_1$. Now modify Ω_2 as below for $0 \leq t \leq t_2$ so that it is Lipschitz for all $t \geq 0$. Since $r^{out}(t_1) \approx r^{out}(t_2) \approx 2$, we can construct $\tilde{\Omega}_2$ satisfying

a. $\tilde{\Omega}_2 \cap \{0 \le t \le t_1\} = B_2(0) \times [0, t_1]$

b. $\tilde{\Omega}_2 \cap \{t_1 \le t \le t_2\} \supset \Omega_2 \cap \{t_1 \le t \le t_2\}$

c. $\tilde{\Omega}_2 \cap \{t_2 \le t < T\} = \Omega_2 \cap \{t_2 \le t < T\}$

d. $\tilde{\Omega}_2$ is Lipschitz in a parabolic scaling.

Let w be a caloric function in $\tilde{\Omega}_2 - \Omega_1$ such that

$$\begin{cases} \Delta w = w_t & \text{in} \quad \tilde{\Omega}_2 - \Omega_1 \\ w = u & \text{on} \quad \{t = 0\} \cup (\partial\Omega_1 \cap \{t > 0\}) \\ w = 0 & \text{on} \quad \partial\tilde{\Omega}_2 \cap \{t > 0\}. \end{cases}$$

For the proof of the lemma, it suffices to prove $u \le w$ since this inequality would imply that the free boundary $\Gamma(u)$ in contained in Ω_2 for $t \ge t_2$, i.e., the outer radius $r^{out}(t) \le (1 + K\alpha)r(t)$ for $t \ge t_2$. Below we prove $u \le w$.

1. Since the free boundary of u shrinks in time,

$$u \le w \text{ for } 0 \le t \le t_1. \tag{4.1}$$

2. Using (3.22) and the Lipschitz property of $\tilde{\Omega}_2$, we will show that

$$u \le w \text{ for } t_1 \le t \le t_4. \tag{4.2}$$

Since $u(\cdot, t_1) \le w(\cdot, t_1)$ (see (4.1)) and $u = w$ on $\partial\Omega_1$, it suffices to prove that w is a supersolution of (P) for $t_1 < t < t_4$. Since $u_0 = \rho \le \alpha$ on $(\tilde{\Omega}_2 - \Omega_1) \cap \{t = 0\}$, the bound (3.22) and the construction of w yield

$$\max w \le C_0 \alpha \tag{4.3}$$

for $C_0 = C_0(n, M)$. On the other hand, since $\tilde{\Omega}_2$ is Lipschitz in a parabolic scaling for $t \ge 0$, $w(\cdot, t)$ is almost harmonic near its vanishing boundary $\partial\tilde{\Omega}_{2t}$ for $t \ge t_1$ (see Lemma 2.1). Observe that for $t_2 \le t \le t_4$

$$\Omega_t(w) = \tilde{\Omega}_{2t} - \Omega_{1t} = B_{(1+K\alpha)r(t)}(0) - B_{r(t)}(0)$$

and for $t_1 \le t \le t_2$

$$\Omega_t(w) \supset B_{(1+K\alpha)r(t)}(0) - B_{r(t)}(0)$$

where $r(t) \approx 1$ for $t_1 \le t \le t_4$ (see (3.2) and (3.12)). Hence we can observe that if $K = K(n, M)$ is chosen sufficiently large, then the almost harmonicity of w with its upper bound (4.3) implies that w is bounded from above by a radial linear function with a small slope so that

$$|\nabla w| < 1 \text{ on } \partial\tilde{\Omega}_2 \cap \{t_1 \le t \le t_4\}. \tag{4.4}$$

Hence w is a supersolution for $t_1 < t < t_4$ and (4.2) follows.

3. Now suppose

$$u \le w \text{ for } 0 \le t \le t_i \tag{4.5}$$

for a fixed $i \ge 4$ and we show

$$u \le w \text{ for } t_i \le t \le t_{i+1}.$$

First, observe that (4.5) implies the free boundary $\Gamma_{t_{i-2}}(u)$ is trapped between

$$\partial\Omega_{1t_{i-2}} = \partial B_{r(t_{i-2})}(0) \text{ and } \partial\tilde{\Omega}_{2t_{i-2}} = \partial B_{(1+K\alpha)r(t_{i-2})}(0).$$

In other words, the inner and outer boundaries of $\tilde{\Omega}_2 - \Omega_1$ at time $t = t_{i-2}$ are located within a distance $K\alpha r(t_{i-2})$ from the free boundary of u. Then since $|\nabla u| \le C_0 M$, we obtain

$$u \le CK\alpha r(t_{i-2}) \text{ on } (\tilde{\Omega}_2 - \Omega_1) \cap \{t = t_{i-2}\} \tag{4.6}$$

for some $C = C(n, M)$. Also by (3.22),

$$u \le C\alpha r(t) \le C\alpha r(t_{i-2}) \text{ on } \partial\Omega_1 \cap \{t_{i-2} \le t \le t_{i+1}\}. \tag{4.7}$$

Now construct two caloric functions w_1 and w_2 in

$$\Pi := (\tilde{\Omega}_2 - \Omega_1) \cap \{t_{i-2} \le t \le t_{i+1}\}$$

such that

$$\begin{cases} \Delta w_1 = \partial w_1/\partial t & \text{in} \quad \Pi \\ w_1 = u & \text{on} \quad \{t = t_{i-2}\} \\ w_1 = 0 & \text{on} \quad \partial\Omega_1 \cup \partial\tilde{\Omega}_2 \end{cases}$$

and that

$$\begin{cases} \Delta w_2 = \partial w_2/\partial t & \text{in} \quad \Pi \\ w_2 = 0 & \text{on} \quad \{t = t_{i-2}\} \cup \partial\tilde{\Omega}_2 \\ w_2 = u & \text{on} \quad \partial\Omega_1. \end{cases}$$

Below we prove

$$u \le w_1 + w_2$$

in $\Pi \cap \{t_i \le t \le t_{i+1}\}$ by showing that $w_1 + w_2$ is a supersolution of (P) for $t_i \le t \le t_{i+1}$. On $\Pi \cap \{t = t_{i-1}\}$,

$$\begin{aligned} w_1 + w_2 &\le \max w_1(\cdot, t_{i-1}) + C\alpha r(t_{i-2}) \\ &\le 2C\alpha r(t_{i-2}) \end{aligned} \tag{4.8}$$

where the first inequality follows from (4.7) and the last inequality follows from (4.6) if $\alpha \le \alpha(n, M)$ for a sufficiently small $\alpha(n, M) > 0$ since the time cross-section of the domain Π, that is $B_{(1+K\alpha)r(t_{i-2})}(0) - B_{r(t_{i-2})}(0)$, has a sufficiently small thickness $K\alpha r(t)$ for $t_{i-2} \le t \le t_{i-1}$. By (4.7) and (4.8),

$$w_1 + w_2 \le C(n, M)\alpha r(t_{i-2})$$

on the parabolic boundary of $\Pi \cap \{t_{i-1} \le t \le t_{i+1}\}$ and hence

$$\max_{\Pi \cap \{t_{i-1} \le t \le t_{i+1}\}} w_1 + w_2 \le C(n, M)\alpha r(t_{i-2}). \tag{4.9}$$

Then by a similar argument as in (4.4) with (4.3) replaced by (4.9), we obtain

$$|\nabla(w_1 + w_2)| < 1$$

on $\partial\tilde{\Omega}_2 \cap \{t_i \le t \le t_{i+1}\}$ if $\alpha \le \alpha(n, M)$ for a sufficiently small $\alpha(n, M) > 0$. Hence we conclude $w_1 + w_2$ is a supersolution of (P) for $t_i \le t \le t_{i+1}$.

By the construction of w_1 and w_2,

$$u = w_1 + w_2$$

on the inner lateral boundary $\partial\Omega_1 \cap \{t_i \le t \le t_{i+1}\}$ of $\Pi \cap \{t_i \le t \le t_{i+1}\}$. Also

$$u(\cdot, t_i) \le (w_1 + w_2)(\cdot, t_i)$$

since the assumption (4.5) implies $\Omega(u) \cap \{t_{i-2} \le t \le t_i\} \subset \tilde{\Omega}_2$. Hence we conclude

$$u \le w_1 + w_2$$

in $\Pi \cap \{t_i \le t \le t_{i+1}\}$. This implies that the free boundary $\Gamma_t(u)$ is contained in $\tilde{\Omega}_2$ for $t_i \le t \le t_{i+1}$. Now recall that the caloric function w has positive set $\tilde{\Omega}_2 - \Omega_1$ with $w = u$ on $\partial\Omega_1$ and $w = 0$ on $\partial\tilde{\Omega}_2$. Also by (4.5), $u \le w$ at time $t = t_i$. Hence by comparison,

$$u \le w \text{ for } t_i \le t \le t_{i+1}.$$

□

5 Interior approximation of u by a radial function ϕ

In this section, we approximate the solution u by a radial function ϕ at interior points, located $\alpha^{2/3}r(t)$-away from $\partial B_{r(t)}(0)$.

• Let ϕ be a radially symmetric function defined in Ω_1 such that on each time interval $[t_i, t_{i+1})$, $\phi(x,t)$ solves

$$\begin{cases} \Delta\phi = \phi_t & \text{in} \quad \Omega_1 \cap \{t_i < t < t_{i+1}\} \\ \phi = 0 & \text{on} \quad \partial\Omega_1 \cap \{t_i < t < t_{i+1}\} \\ \phi(x,t_i) = \phi(|x|,t_i) = \min\limits_{\{y:|y|=|x|\}} u(y,t_i) & \text{on} \quad \Omega_1 \cap \{t = t_i\}. \end{cases}$$

In other words,

(i) $\phi(\cdot, t_i)$ is the maximal radial function $\leq u$.

(ii) $\phi(x,t)$ is caloric in $\Omega_1 \cap \{t_i < t < t_{i+1}\}$ with $\phi = 0$ on $\partial\Omega_1 \cap \{t_i < t < t_{i+1}\}$, and $\phi = \phi(\cdot, t_i)$ on $\Omega_1 \cap \{t = t_i\}$.

Note that ϕ need not be continuous at $t = t_i$. By comparison, $\phi \leq u$.

Lemma 5.1. *Let u_0 be as in Theorem 1.1 and let $\alpha \leq \alpha(n,M)$ for a sufficiently small $\alpha(n,M) > 0$. Let $\epsilon = 2/3$ and let $t \in [t_2, T)$. Then on $B_{(1-\alpha^\epsilon)r(t)}(0)$*

$$\phi(\cdot,t) \leq u(\cdot,t) \leq (1 + C\alpha^{1-\epsilon})\phi(\cdot,t) \tag{5.1}$$

for a constant $C > 0$ depending on n and M.

Proof. Fix $\tau \in [t_i, t_{i+1})$ for $i \geq 2$. Let w solve

$$\begin{cases} \Delta w = w_t & \text{in} \quad \Omega_1 \cap \{t_{i-1} < t < \tau\} \\ w = 0 & \text{on} \quad \partial\Omega_1 \cap \{t_{i-1} < t < \tau\} \\ w = \phi & \text{on} \quad \Omega_1 \cap \{t = t_{i-1}\}. \end{cases}$$

Observe that w is a radially symmetric function with $w \leq u$. Since $\phi(\cdot, t_i)$ is the maximal radial function $\leq u(\cdot, t_i)$, we obtain

$$w(\cdot, t_i) \leq \phi(\cdot, t_i).$$

Then by comparison

$$w \leq \phi \ \text{ for } \ t_{i-1} \leq t \leq \tau. \tag{5.2}$$

On the other hand, by a similar argument as in the proof of (3.15) we obtain

$$u(\cdot, t_i) - \phi(\cdot, t_i) \le C\alpha r(t_i) \tag{5.3}$$

for $C = C(n, M) > 0$ and $i \in \mathbb{N}$ since u is periodic in angle with period $\le \alpha$, $|\nabla u| \le C_0 M$ and $\phi(\cdot, t_i)$ is the maximal radial function $\le u(\cdot, t_i)$. Hence for some $C = C(n, M) > 0$

$$\begin{aligned} Cr(t_{i-1}) &\le \max u(\cdot, t_{i-1}) \\ &\le 2\max \phi(\cdot, t_{i-1}) = 2\max w(\cdot, t_{i-1}) \end{aligned}$$

where the first inequality follows from Lemma 3.1 and the second inequality follows from (5.3). The above inequality implies that for $C = C(n, M) > 0$

$$Cr(\tau) \le \max w(\cdot, \tau) \tag{5.4}$$

since $r(t_{i-1}) \approx r(t)$ for $t_{i-1} \le t \le \tau$ (Lemma 3.1), $|\nabla u| \le C_0 M$ (Lemma 2.5) and Ω_1 is Lipschitz in a parabolic scaling. The Lipschitz property of $\Omega_1 \cap \{t_{i-1} \le t \le \tau\}$ implies that $w(\cdot, \tau)$ is almost harmonic near $\partial\Omega_{1\tau}$ (Lemma 2.1). Then by a similar reasoning as in (4.4) with the lower bound (5.4), we obtain that $w(|x|, \tau)$ is bounded from below by a radial linear function vanishing on $|x| = r(\tau)$, with slope $c = c(n, M)$. Hence on $B_{(1-\alpha^\epsilon)r(\tau)}(0)$

$$c(n, M)\alpha^\epsilon r(\tau) \le w(\cdot, \tau) \le \phi(\cdot, \tau) \tag{5.5}$$

where the last inequality follows from (5.2).

Now observe that by (5.3)

$$u(\cdot, t_i) - \phi(\cdot, t_i) \le C\alpha r(t_i)$$

and on $\partial\Omega_1 \cap \{t_i < t < \tau\}$

$$u - \phi = u \le C\alpha r(t_i)$$

where the inequality follows from (3.22) and Lemma 3.1. Hence by comparison

$$u(\cdot, t) - \phi(\cdot, t) \le C\alpha r(t_i) \tag{5.6}$$

for $t_i \le t \le \tau$. Conclude that on $B_{(1-\alpha^\epsilon)r(\tau)}(0)$

$$\begin{aligned} u(\cdot, \tau) - \phi(\cdot, \tau) &\le C\alpha r(t_i) \\ &\le C\alpha r(\tau) \\ &\le C\alpha^{1-\epsilon}\phi(\cdot, \tau) \end{aligned}$$

for $C = C(n, M) > 0$ where the first inequality follows from (5.6), the second inequality follows from Lemma 3.1 and the last inequality follows from (5.5). □

6 Interior improvement by flatness of boundary

We improve the interior estimate in Lemma 5.1, using the flatness of the free boundary, that is Lemma 4.1. More precisely, the constant C in the interior estimate (5.1) improves in time, up to an order determined by the 'flatness constant' of the free boundary. Let u_0 be as in Theorem 1.1 and let $\alpha \le \alpha(n, M)$ for a sufficiently small $\alpha(n, M) > 0$, throughout the rest of the paper.

• Let $\tilde{\phi}$ be a radially symmetric function defined in Ω_1 such that

$$\tilde{\phi}(x,t) = \tilde{\phi}(|x|,t) = \min_{\{y:|y|=|x|\}} u(y,t).$$

In other words, $\tilde{\phi}$ is the maximal radial function in Ω_1 such that $\tilde{\phi} \le u$. Note that $\phi(\cdot, t_i) = \tilde{\phi}(\cdot, t_i)$ for $i \in \mathbb{N}$, and $\phi \le \tilde{\phi}$ since ϕ is radial with $\phi \le u$.

Lemma 6.1. *Let $\epsilon = 2/3$. Assume*

(a) (5.1) holds at time $t = t_i$, i.e., on $B_{(1-\alpha^\epsilon)r(t_i)}(0)$

$$u(\cdot, t_i) \le (1 + C\alpha^{1-\epsilon})\phi(\cdot, t_i)$$

(b) For $t_i \le t \le t_{i+1}$, $\Gamma_t(u)$ is contained in $B_{(1+K\alpha)r(t)}(0) - B_{r(t)}(0)$ for some constant K satisfying

$$1 \le K < \alpha^{\frac{\epsilon-1}{2}} C \tag{6.1}$$

(c) on $B_{r(t_i)}(0) - B_{(1-\alpha^\epsilon)r(t_i)}(0)$,

$$u(\cdot, t_i) \le \phi(\cdot, t_i) + L(C + K)\alpha r(t_i)$$

where L is a positive constant depending on n and M; C and K are the same constants as in (a) and (b). Then for a constant $0 < h = h(n, M) < 1$, the condition (a) holds with C replaced by hC at time $t = t_{i+1}$, i.e.,

$$u(\cdot, t_{i+1}) \le (1 + hC\alpha^{1-\epsilon})\phi(\cdot, t_{i+1}) \tag{6.2}$$

on $B_{(1-\alpha^\epsilon)r(t_{i+1})}(0)$. If we further assume

$$u(x,t) \le (1 + C\alpha^{1-\epsilon})\tilde{\phi}(x,t) \tag{6.3}$$

for $t_i \le t \le t_{i+1}$ and $x \in B_{(1-\alpha^\epsilon)r(t)}(0)$, then the condition (c) holds with C replaced by hC at time $t = t_{i+1}$, i.e.,

$$u(\cdot, t_{i+1}) \le \phi(\cdot, t_{i+1}) + L(hC + K)\alpha r(t_{i+1}) \tag{6.4}$$

on $B_{r(t_{i+1})}(0) - B_{(1-\alpha^\epsilon)r(t_{i+1})}(0)$.

Proof. For the proof of (6.2), construct caloric functions ψ_1, ψ_2, ψ_3 and ψ_4 in $\Sigma := \Omega_1 \cap \{t_i < t < t_{i+1}\}$ with the following boundary values

$$\begin{cases} \psi_1 = \phi & \text{on} \quad B_{r(t_i)}(0) \times \{t = t_i\} \\ \psi_1 = 0 & \text{otherwise on} \quad \partial\Sigma \end{cases}$$

$$\begin{cases} \psi_2 = u - \phi & \text{on} \quad B_{(1-\alpha^\epsilon)r(t_i)}(0) \times \{t = t_i\} \\ \psi_2 = 0 & \text{otherwise on} \quad \partial\Sigma \end{cases}$$

$$\begin{cases} \psi_3 = u - \phi & \text{on} \quad B_{r(t_i)}(0) - B_{(1-\alpha^\epsilon)r(t_i)}(0) \times \{t = t_i\} \\ \psi_3 = 0 & \text{otherwise on} \quad \partial\Sigma \end{cases}$$

$$\begin{cases} \psi_4 = u & \text{on} \quad \partial\Sigma \cap \{t_i < t < t_{i+1}\} \\ \psi_4 = 0 & \text{otherwise on} \quad \partial\Sigma. \end{cases}$$

Then in Σ

$$u = \psi_1 + \psi_2 + \psi_3 + \psi_4$$

where ψ_1 is radially symmetric since ϕ is radially symmetric and Ω_1 is radial. For $j = 2, 3, 4$, let $\psi_{j1}(\cdot)$ be the maximal radial function on $B_{r(t_{i+1})}(0)$ such that $\psi_{j1}(\cdot) \le \psi_j(\cdot, t_{i+1})$ and write $\psi_j(\cdot, t_{i+1}) = \psi_{j1}(\cdot) + \psi_{j2}(\cdot)$. Since $\phi(\cdot, t_{i+1})$ is the maximal radial function $\le u(\cdot, t_{i+1})$,

$$\phi(\cdot, t_{i+1}) \ge \psi_1(\cdot, t_{i+1}) + \psi_{21}(\cdot) + \psi_{31}(\cdot) + \psi_{41}(\cdot).$$

Hence for (6.2), it suffices to prove

$$\psi_{22}(\cdot) + \psi_{32}(\cdot) + \psi_{42}(\cdot) \le hC\alpha^{1-\epsilon}\phi(\cdot, t_{i+1}).$$

1. First, we prove that for some constant $0 < h_0 = h_0(n, M) < 1$

$$\psi_{22}(\cdot) \le h_0 C\alpha^{1-\epsilon}\phi(\cdot, t_{i+1}). \tag{6.5}$$

Suppose that at some $x \in B_{r(t_{i+1})}(0)$

$$\psi_2(x, t_{i+1}) > \frac{1}{2}C\alpha^{1-\epsilon}\psi_1(x, t_{i+1}). \tag{6.6}$$

(Otherwise, (6.5) would hold with $h_0 = 1/2$ since $\psi_1 \le \phi$ and $\psi_{22} \le \psi_2$.) Since Σ is Lipschitz in a parabolic scaling (Lemma 3.2), Lemma 2.1 imply that near the vanishing boundary $\partial B_{r(t_{i+1})}(0)$, $\psi_1(\cdot, t_{i+1})$ and $\psi_2(\cdot, t_{i+1})$ are comparable to some harmonic functions vanishing on $\partial B_{r(t_{i+1})}(0)$. Hence (6.6) implies that on $B_{r(t_{i+1})}(0)$

$$\psi_2(\cdot, t_{i+1}) \geq h_1 C\alpha^{1-\epsilon}\psi_1(\cdot, t_{i+1})$$

for some constant $0 < h_1 = h_1(n, M) < 1$. Since ψ_1 is radially symmetric and ψ_{21} is the maximal radial function $\leq \psi_2$, the above inequality implies

$$\psi_{21} \geq h_1 C\alpha^{1-\epsilon}\psi_1(\cdot, t_{i+1}). \tag{6.7}$$

Let $h_0 = 1 - h_1$, then on $B_{r(t_{i+1})}(0)$

$$\begin{aligned}
\psi_{22}(\cdot) = \psi_2(\cdot, t_{i+1}) - \psi_{21}(\cdot) &\leq C\alpha^{1-\epsilon}\psi_1(\cdot, t_{i+1}) - \psi_{21}(\cdot) \\
&\leq (1 - h_1)C\alpha^{1-\epsilon}\psi_1(\cdot, t_{i+1}) \\
&\leq h_0 C\alpha^{1-\epsilon}\phi(\cdot, t_{i+1})
\end{aligned}$$

where the first inequality follows from the assumption (a) with the construction of ψ_1 and ψ_2, and the second inequality follows from (6.7). Hence we obtain the upper bound (6.5) of ψ_{22}.

2. Next we show

$$\psi_{32}(\cdot) \leq \frac{1 - h_0}{3} C\alpha^{1-\epsilon}\phi(\cdot, t_{i+1}). \tag{6.8}$$

By the assumption (c) with the construction of ψ_3,

$$\max \psi_3(\cdot, t_i) \leq L(C + K)\alpha r(t_i)$$

on the annulus $R := B_{r(t_i)}(0) - B_{(1-\alpha^\epsilon)r(t_i)}(0)$. Let $|R|$ denote the volume of R, then $|R| \approx \alpha^\epsilon |B_{r(t_i)}(0)|$. Hence there exists a small constant $c(\alpha^\epsilon, n) > 0$ such that

$$\max \psi_3(\cdot, t_{i+1}) \leq c(\alpha^\epsilon, n)L(C + K)\alpha r(t_i)$$

where $c(\alpha^\epsilon, n) \to 0$ as $\alpha \to 0$. Since $r(t_i) \approx r(t_{i+1}) \approx \max \psi_1(\cdot, t_{i+1})$ (Lemma 3.1),

$$\max \psi_3(\cdot, t_{i+1}) \leq c(\alpha^\epsilon, n)C_0(n, M)L(C + K)\alpha \max \psi_1(\cdot, t_{i+1}) \tag{6.9}$$

for some $C_0(n, M) > 0$. Hence on $B_{r(t_{i+1})}(0)$

$$\begin{aligned}
\psi_{32}(\cdot) \leq \psi_3(\cdot, t_{i+1}) &\leq c(\alpha^\epsilon, n)C_1(n, M)L(C + K)\alpha\psi_1(\cdot, t_{i+1}) \\
&\leq c(\alpha^\epsilon, n)C_1(n, M)L(C + K)\alpha\phi(\cdot, t_{i+1}) \\
&\leq c(\alpha^\epsilon, n)C_2(n, M)LC\alpha^{\frac{\epsilon+1}{2}}\phi(\cdot, t_{i+1}) \\
&\leq \frac{1 - h_0}{3} C\alpha^{1-\epsilon}\phi(\cdot, t_{i+1})
\end{aligned} \tag{6.10}$$

where $C_1(n,M)$ and $C_2(n,M)$ are positive constants depending on n and M, the first inequality follows from the almost harmonicity of $\psi_3(\cdot,t_{i+1})$ and $\psi_1(\cdot,t_{i+1})$ with (6.9), the second inequality follows since $\psi_1 \leq \phi$, the third inequality follows from (6.1), and the last inequality follows if $\alpha < \alpha(n,M)$ for a sufficiently small $\alpha(n,M) > 0$ (since $\epsilon = 2/3$ and L and h_0 are constants depending on n and M).

3. Since $\psi_{42}(\cdot) = \psi_4(\cdot,t_{i+1}) - \psi_{41}(\cdot)$ where $\psi_{41}(\cdot)$ is the maximal radial function $\leq \psi_4(\cdot,t_{i+1})$,

$$\begin{aligned}\max_{\partial B_s(0)} \psi_{42}(\cdot) &= \max_{\partial B_s(0)} \psi_4(\cdot,t_{i+1}) - \psi_{41}|_{\partial B_s(0)} \\ &= \max_{\partial B_s(0)} \psi_4(\cdot,t_{i+1}) - \min_{\partial B_s(0)} \psi_4(\cdot,t_{i+1}) \qquad (6.11)\end{aligned}$$

for $0 \leq s \leq r(t_{i+1})$. Below we prove that the right hand side of (6.11) is bounded from above by

$$\frac{1-h_0}{3} C\alpha^{1-\epsilon}\phi(\cdot,t_{i+1})$$

if $0 \leq s \leq (1-\alpha^\epsilon)r(t_{i+1})$.

Let $0 < s \leq (1-\alpha^\epsilon)r(t_{i+1})$. Let x_1 be a maximum point of ψ_4 on $\partial B_s(0)$, i.e.,

$$\max_{\partial B_s(0)} \psi_4(\cdot,t_{i+1}) = \psi_4(x_1,t_{i+1}).$$

Since ψ_4 is periodic in angle with period $\leq \alpha$, there exists a minimum point x_2 of ψ_4 on $\partial B_s(0)$ such that

$$\min_{\partial B_s(0)} \psi_4(\cdot,t_{i+1}) = \psi_4(x_2,t_{i+1})$$

and

$$|x_1 - x_2| < C_n \alpha r(t_{i+1})$$

where C_n is a dimensional constant. Recall that ψ_4 is a nonnegative caloric function in Σ with

$$\max_{\Sigma} \psi_4 = \max_{\partial\Omega_1 \cap \{t_i < t < t_{i+1}\}} u \leq C(n,M)\alpha r(t_i) \leq C(n,M)K\alpha r(t_i) \qquad (6.12)$$

where the first inequality follows from (3.22) and the last inequality follows if $K \geq 1$. Now apply Lemma 2.4 for ψ_4 with $\delta = \alpha^\epsilon r(t_{i+1})$ and $\sigma = C_n\alpha r(t_{i+1})/\delta$. Then the upper bound (6.12) of ψ_4 implies that on $Q^-_{\sigma\delta}(x_1,t_{i+1})$

$$|\nabla\psi_4| \leq \frac{C(n,M)K\alpha r(t_i)}{\alpha^\epsilon r(t_{i+1})}.$$

Since $|x_1 - x_2| < C_n\alpha r(t_{i+1}) = \sigma\delta$, the above bound on $|\nabla\psi_4|$ yields that

$$\begin{aligned}
\max_{\partial B_s(0)} \psi_4(\cdot, t_{i+1}) - \min_{\partial B_s(0)} \psi_4(\cdot, t_{i+1}) &= \psi_4(x_1, t_{i+1}) - \psi_4(x_2, t_{i+1}) \\
&\leq C(n, M)K\alpha^{2-\epsilon}r(t_i) \\
&\leq C_1K\alpha^{2-\epsilon}r(t_{i+1}) \qquad (6.13)
\end{aligned}$$

for a constant C_1 depending on n and M. Suppose $\alpha < \alpha(n, M)$ for a sufficiently small constant $\alpha(n, M) > 0$, then we can bound the right hand side of (6.13) as in below

$$\begin{aligned}
C_1K\alpha^{2-\epsilon}r(t_{i+1}) &\leq C_1K\alpha^{2-2\epsilon}\min\{\phi(x, t_{i+1}) : x \in B_{(1-\alpha^\epsilon)r(t_{i+1})}(0)\} \\
&\leq C_1C\alpha^{3(1-\epsilon)/2}\min\{\phi(x, t_{i+1}) : x \in B_{(1-\alpha^\epsilon)r(t_{i+1})}(0)\} \\
&\leq \frac{1-h_0}{3}C\alpha^{1-\epsilon}\min\{\phi(\cdot, t_{i+1}) : x \in B_{(1-\alpha^\epsilon)r(t_{i+1})}(0)\}
\end{aligned}$$

where C_1 denotes a constant depending on n and M different in places, the first inequality follows from (5.5), the second inequality follows from (6.1), and the last inequality follows if $\alpha(n, M)$ is sufficiently small since $\epsilon = 2/3$ and h_0 is a constant depending on n and M. Combining the above inequality with (6.11) and (6.13) we obtain that

$$\psi_{42}(\cdot, t_{i+1}) \leq \frac{1-h_0}{3}C\alpha^{1-\epsilon}\phi(\cdot, t_{i+1}) \qquad (6.14)$$

on $B_{(1-\alpha^\epsilon)r(t_{i+1})}(0)$.

From the upper bounds (6.5), (6.8) and (6.14) on ψ_{22}, ψ_{32} and ψ_{42}, we conclude

$$\psi_{22} + \psi_{32} + \psi_{42} \leq \frac{h_0+2}{3}C\alpha^{1-\epsilon}\phi(\cdot, t_{i+1})$$

on $B_{(1-\alpha^\epsilon)r(t_{i+1})}(0)$. Hence the first part of the lemma, that is (6.2), follows for the constant $h := (h_0 + 2)/3 < 1$.

Next we prove the second part of the lemma, that is (6.4). Let Π be a thin subregion of $\Omega_1 \cap \{t_i \leq t \leq t_{i+1}\}$ such that

$$\Pi_t = B_{r(t)}(0) - B_{(1-\alpha^\epsilon)r(t)}(0)$$

for $t_i \leq t \leq t_{i+1}$. Decompose u into a sum of three caloric functions w_1, w_2 and

w_3, which are defined in Π with the following boundary conditions

$$\begin{cases} w_1(\cdot,t) = u(\cdot,t) & \text{on } \partial B_{(1-\alpha^\epsilon)r(t)}(0) \text{ for } t_i \le t \le t_{i+1} \\ w_1 = 0 & \text{otherwise on } \partial\Pi \end{cases}$$

$$\begin{cases} w_2(\cdot,t) = u(\cdot,t) & \text{on } \partial B_{r(t)}(0) \text{ for } t_i \le t \le t_{i+1} \\ w_2 = 0 & \text{otherwise on } \partial\Pi \end{cases}$$

$$\begin{cases} w_3(\cdot,t_i) = u(\cdot,t_i) & \text{on } B_{r(t_i)}(0) - B_{(1-\alpha^\epsilon)r(t_i)}(0) \\ w_3 = 0 & \text{otherwise on } \partial\Pi. \end{cases}$$

Observe $u = w_1 + w_2 + w_3$ in Π. Let w_{11} be the maximal radial function such that $w_{11}(\cdot) \le w_1(\cdot, t_{i+1})$ and let w_{31} be the maximal radial function such that $w_{31}(\cdot) \le w_3(\cdot, t_{i+1})$. Then on $B_{r(t_{i+1})}(0) - B_{(1-\alpha^\epsilon)r(t_{i+1})}(0) \times \{t = t_{i+1}\}$

$$u - \phi \le (w_1 - w_{11}) + w_2 + (w_3 - w_{31})$$

since $\phi(\cdot, t_{i+1})$ is the maximal radial function $\le u(\cdot, t_{i+1})$. Hence for (6.4), it suffices to prove that the right hand side of the above inequality is bounded by $L(hC + K)\alpha r(t_{i+1})$.

1. Bound on $w_1 - w_{11}$:

$$\begin{aligned} w_1(\cdot, t_{i+1}) - w_{11}(\cdot) &\le C\alpha^{1-\epsilon} w_{11}(\cdot) \\ &\le C\alpha^{1-\epsilon} C_1(n,M)\alpha^\epsilon r(t_{i+1}) \\ &\le \frac{hCL}{2}\alpha r(t_{i+1}) \end{aligned}$$

where the first inequality follows from the assumption (6.3) since

$$w_1 = u \le \tilde{\phi} + C\alpha^{1-\epsilon}\tilde{\phi}$$

on $\partial B_{(1-\alpha^\epsilon)r(t)}(0) \times \{t\}$ and w_{11} is bounded below by a radial caloric function in Π with boundary value $\tilde{\phi}$ on $\partial B_{(1-\alpha^\epsilon)r(t)}(0) \times \{t\}$, the second inequality follows from $\max w_{11} \le \max_{\Pi} u \le C_1(n,M)\alpha^\epsilon r(t_{i+1})$, and the last inequality holds if $L = L(n,M)$ is a sufficiently large constant depending on n an M.

2. Bound on w_2: For $t_i \le t \le t_{i+1}$,

$$\max_{\partial B_{r(t)}(0)} u(\cdot,t) \le C(n,M)\alpha r(t) \le C(n,M)K\alpha r(t) \tag{6.15}$$

where the first inequality follows from (3.22) and the last inequality follows if $K \geq 1$. Hence from the construction of w_2,

$$\begin{aligned} \max w_2(\cdot, t_{i+1}) &\leq C(n, M)K\alpha r(t_{i+1}) \\ &\leq \frac{LK}{2}\alpha r(t_{i+1}) \end{aligned}$$

where the last inequality follows if $L = L(n, M)$ is sufficiently large.

3. Bound on $w_3 - w_{31}$: A similar argument as in (6.9) shows that (c) implies

$$w_3(\cdot, t_{i+1}) - w_{31}(\cdot) \leq c(\alpha^\epsilon, n)C(n, M)L(C + K)\alpha r(t_{i+1})$$

where $c(\alpha^\epsilon, n) \to 0$ as $\alpha \to 0$. Hence if $\alpha < \alpha(n, M)$ for a sufficiently small $\alpha(n, M)$, then

$$w_3(\cdot, t_{i+1}) - w_{31}(\cdot) \leq \frac{hL(C + K)}{2}\alpha r(t_{i+1}).$$

Combing the above bounds on $w_1 - w_{11}$, w_2, and $w_3 - w_{31}$, we conclude

$$\begin{aligned} u(\cdot, t_{i+1}) - \phi(\cdot, t_{i+1}) &\leq w_1(\cdot, t_{i+1}) - w_{11} + w_2(\cdot, t_{i+1}) + w_3(\cdot, t_{i+1}) - w_{31} \\ &\leq L(hC + K)\alpha r(t_{i+1}) \end{aligned}$$

□

Remark 4. In Lemma 6.1, we assume $K \geq 1$ for the simplicity of proof. In the proof of Lemma 6.1, this assumption is used only for (6.12) and (6.15), i.e., for a bound on $\max_{\partial\Omega_1} u$. Later, to iterate Lemma 6.1 and Lemma 7.1 for a decreasing sequence of $K < 1$, we will modify and improve the inner region Ω_1 so that (6.12) and (6.15) are guaranteed for $K < 1$ (see Corollary 7.2).

Corollary 6.2. *Let $0 < h < 1$ and $\epsilon = 2/3$ be as in Lemma 6.1. Let m be the largest integer satisfying*

$$1 < \alpha^{\frac{\epsilon - 1}{2}} h^m.$$

Then for $j \geq m + 2$, (a) and (c) of Lemma 6.1 hold with C replaced by $\alpha^{\frac{1-\epsilon}{2}} C$ at time $t = t_j$. Here $C > 0$ is a constant depending on n and M.

Proof. By Lemma 4.1 and Lemma 5.1, the conditions (a) and (b) of Lemma 6.1 are satisfied with $C = K = C(n, M) \geq 1$ for $i \geq 2$. The condition (c) of Lemma 6.1 follows from (5.6) for $i \geq 2$. Also the condition (6.3) holds for $t \geq t_2$ by Lemma 5.1 with $\phi \leq \tilde{\phi}$. Hence applying Lemma 6.1, we obtain that the conditions (a) and (c) hold with C replaced by hC for $i \geq 3$.

On the other hand, the inequality (6.1) of the condition (b) holds for the constants K and hC if $\alpha < \alpha(n,M)$ for a sufficiently small $\alpha(n,M)$ so that $K = C < \alpha^{\frac{\epsilon-1}{2}} hC$. Hence the condition (b) holds for $i \geq 3$ with the improved constant hC.

Next we verify the condition (6.3) with C replaced by hC for $t \geq t_3$. Fix $\tau \in (t_i, t_{i+1})$ for $i \geq 3$. Decompose the time interval $(0,T)$ so that

$$0 < s_1 = t_1 < ... < s_{i-1} = t_{i-1} < s_i = \tau < s_{i+1} = t_{i+2} < ... < T.$$

Then by a similar argument as in the proof of (6.2) of Lemma 6.1,

$$u(\cdot,\tau) \leq (1 + hC\alpha^{1-\epsilon})\tilde{\phi}(\cdot,\tau) \quad \text{on} \quad B_{(1-\alpha^\epsilon)r(\tau)}(0).$$

Hence the condition (6.3) is satisfied for $t \geq t_3$ with C replaced by hC. Then applying Lemma 6.1 for $i \geq 3$, we obtain a better constant h^2C for the conditions (a) and (c) for $i \geq 4$.

Now let m be the largest integer satisfying

$$1 < \alpha^{\frac{\epsilon-1}{2}} h^m.$$

Then the inequality (6.1) holds with C replaces by h^jC for $1 \leq j \leq m$. Hence we can iterate Lemma 6.1 m times, as above, and obtain the improved constant $\alpha^{\frac{1-\epsilon}{2}}C$ in (a) and (c) for $i \geq m+2$. In other words, for $i \geq m+2$

$$u(\cdot,t_i) \leq (1 + \alpha^{\frac{1-\epsilon}{2}} C\alpha^{1-\epsilon})\phi(\cdot,t_i)$$

on $B_{(1-\alpha^\epsilon)r(t_i)}(0)$ and

$$u(\cdot,t_i) \leq \phi(\cdot,t_i) + L(\alpha^{\frac{1-\epsilon}{2}} C + K)\alpha r(t_i)$$

on $B_{r(t_i)}(0) - B_{(1-\alpha^\epsilon)r(t_i)}(0)$. □

7 Improvement on flatness by interior estimate; Asymptotic behavior of free boundary

In this section we show that the improved interior estimate, as in Corollary 6.2, propagates to the free boundary at later times and gives an improved estimate on the location of the free boundary. More precisely, we improve the constant K in condition (b) of Lemma 6.1, using the improved constants in the conditions (a) and (c).

Lemma 7.1. *Suppose that (a), (b) and (c) of Lemma 6.1 hold for $i \geq i_0$ with $C = \beta$, a small constant. Then for $i \geq i_0 + 2$, (b) and (c) holds with K replaced by $C_1\beta$, for a constant $C_1 > 0$ depending on n and M. In other words, $\Gamma_t(u)$ is contained in $B_{(1+C_1\beta\alpha)r(t)}(0) - B_{r(t)}(0)$ for $t \geq t_{i_0+2}$, and*

$$u(\cdot, t_i) \leq \phi(\cdot, t_i) + L(\beta + C_1\beta)\alpha r(t_i)$$

on $B_{r(t_i)}(0) - B_{(1-\alpha^\epsilon)r(t_i)}(0)$ for $i \geq i_0 + 2$.

Proof. To prove the lemma at $t = t_{i_0+2}$, we will construct a radially symmetric caloric function $w \leq u$ and a radially symmetric supercaloric function $v \geq u$ such that their free boundaries $\Gamma_{t_{i_0+2}}(w)$ and $\Gamma_{t_{i_0+2}}(v)$ are located in the $C_1\beta\alpha r(t_{i_0+2})$-neighborhood of each other. Recall that u is well approximated by a radial function ϕ on each dyadic time interval. However the function ϕ (or w_1 which will be constructed below) does not catch up the change in values of u caused by the displacement of the free boundary from $\partial\Omega_1$. Hence we modify the approximating function w_1 by adding an auxiliary function w_2, and construct two caloric functions $w \leq u$ and $v \geq u$ using the modified approximation, $w_1 + w_2$, of u.

Let w_1 solve

$$\begin{cases} \Delta w_1 = \partial w_1/\partial t & \text{in} \quad \Omega_1 \cap \{t_{i_0} < t < t_{i_0+2}\} \\ w_1 = \phi & \text{on} \quad \Omega_1 \cap \{t = t_{i_0}\} \\ w_1 = 0 & \text{on} \quad \partial\Omega_1 \cap \{t_{i_0} < t < t_{i_0+2}\}. \end{cases}$$

Note that $w_1 = \phi$ for $t_{i_0} \leq t < t_{i_0+1}$ and $w_1 \leq \phi$ for $t_{i_0+1} \leq t \leq t_{i_0+2}$. Recall that ψ_i $(1 \leq i \leq 4)$ are the caloric functions constructed in the proof of Lemma 6.1. Let $\tilde{\psi}_4$ solve

$$\begin{cases} \Delta \tilde{\psi}_4 = \partial \tilde{\psi}_4/\partial t & \text{in} \quad \Omega_1 \cap \{t_{i_0} < t < t_{i_0+2}\} \\ \tilde{\psi}_4 = 0 & \text{on} \quad \Omega_1 \cap \{t = t_{i_0}\} \\ \tilde{\psi}_4 = u & \text{on} \quad \partial\Omega_1 \cap \{t_{i_0} < t < t_{i_0+2}\}. \end{cases}$$

Note that $\tilde{\psi}_4 = \psi_4$ for $t_{i_0} \leq t < t_{i_0+1}$ and $\tilde{\psi}_4 \geq \psi_4$ for $t_{i_0+1} \leq t \leq t_{i_0+2}$. Let Σ' be a space time region in $\Omega_1 \cap \{t_{i_0} \leq t \leq t_{i_0+2}\}$ such that its time cross-section

$$\Sigma'_t = B_{r(t)}(0) - B_{(1-\alpha^\epsilon)r(t)}(0)$$

for $t_{i_0} \leq t \leq t_{i_0+2}$. Let w_2 solve

$$\begin{cases} \Delta w_2 = \partial w_2/\partial t & \text{in} \quad \Sigma' \\ w_2 = 0 & \text{on} \quad \{t = t_{i_0}\} \cup \partial\Omega_1 \\ w_2 = \tilde{\psi}_4 & \text{on} \quad \partial\Sigma' - \{t = t_{i_0}\} - \partial\Omega_1. \end{cases}$$

Note that w_2 has a nonzero boundary values $\tilde{\psi}_4(\cdot, t)$ only on the inner boundary $\partial B_{(1-\alpha^\epsilon)r(t)}(0) \times \{t\}$ of Σ'. Now let

$$w = w_1 + w_2 \text{ in } \Sigma'.$$

Then $w = \phi \leq u$ on $\{t = t_{i_0}\}$ and $w = w_1 + \tilde{\psi}_4 \leq u$ on $\partial B_{(1-\alpha^\epsilon)r(t)}(0) \times \{t\}$ since $w_1 + \tilde{\psi}_4$ is caloric in $\Omega_1 \cap \{t_{i_0} < t < t_{i_0+2}\}$ with $w_1 + \tilde{\psi}_4 \leq u$ on $\partial\Omega_1 \cup \{t = t_{i_0}\}$. Hence by comparison

$$w \leq u \text{ in } \Sigma'.$$

Next to construct a supercaloric function $v \geq u$, we modify the boundary of w on the time interval $[t_{i_0+1}, t_{i_0+2}]$ and also modify the values of w in the new region so that it is a supersolution of (P) with larger boundary values than u. Let $f(t)$ be the linear function defined on the interval $[t_{i_0+1}, t_{i_0+2}]$ such that

$$\begin{cases} f(t_{i_0+1}) = 1 - C_1 K\alpha \\ f(t_{i_0+2}) = 1 - 2C_1\beta\alpha \end{cases}$$

where $C_1 = C_1(n, M)$ is a sufficiently large constant which will be determined later. Here we assume $K > 2\beta$ since otherwise the lemma would hold with $C_1 = 2$. For a fixed $t \in [t_{i_0+1}, t_{i_0+2}]$, let $g(x,t)$ be the harmonic function defined in $B_{r(t)/f(t)}(0) - B_{(1-\alpha^\epsilon)r(t)/f(t)}(0)$ such that

$$\begin{cases} g(x,t) = 1 & \text{for} \quad x \in \partial B_{r(t)/f(t)}(0) \\ g(x,t) = 1 - C_1 K\alpha & \text{for} \quad x \in \partial B_{(1-\alpha^\epsilon)r(t)/f(t)}(0). \end{cases}$$

Let Π be a space time region constructed on the time interval $[t_{i_0+1}, t_{i_0+2}]$ such that its time cross-section

$$\Pi_t = B_{r(t)/f(t)}(0) - B_{(1-\alpha^\epsilon)r(t)/f(t)}$$

for $t \in [t_{i_0+1}, t_{t+2}]$. Now construct a function v in Π as follows

$$v(x,t) = g(x,t)w(f(t)x,t).$$

We will show that v is a supersolution of (P) satisfying $v \geq u$ on the parabolic boundary of Π.

1. To prove that v is supercaloric in Π, we find some bounds on $|f'(t)|$, $|g_t|$, $|\nabla g|$, w, $|\nabla w|$ and $|w_t|$.

(1) Since f is linear and $t_{i_0+2} - t_{i_0+1} \approx r^2(t_{i_0+1})$ (Lemma 3.1)

$$|f'(t)| \leq \frac{C(n,M)C_1K\alpha}{r^2(t_{i_0+1})}. \tag{7.1}$$

(2) Since $g(\cdot,t)$ is harmonic on the annulus $B_{r(t)/f(t)}(0) - B_{(1-\alpha^\epsilon)r(t)/f(t)}(0)$

$$\frac{c(n)C_1K\alpha}{\alpha^e r(t)} \leq |\nabla g| \leq \frac{C(n)C_1K\alpha}{\alpha^e r(t)}. \tag{7.2}$$

(3) From the construction of g

$$\begin{aligned}
|g_t| &\leq \max|\nabla g|\left(\frac{d}{dt}\frac{r(t)}{f(t)} + r'(t)\right) \\
&\leq \max|\nabla g| C(n,M)\left(\frac{C_1K\alpha}{r(t_{i_0+1})} + r'(t)\right) \\
&\leq \max|\nabla g| C(n,M)\frac{C_1K\alpha+1}{r(t_{i_0+1})} \\
&\leq \frac{C(n,M)C_1K\alpha(C_1K\alpha+1)}{\alpha^e r^2(t_{i_0+1})}
\end{aligned} \tag{7.3}$$

where the second inequality follows from (7.1), the third inequality follows from the Lipschitz property of Ω_1 (Lemma 3.2) with Lemma 3.1, and the last inequality follows from (7.2) with Lemma 3.1.

(4) Since $\max\phi(\cdot,t) \approx \max u(\cdot,t) \approx r(t)$,

$$\max w_1(\cdot,t) \approx r(t). \tag{7.4}$$

(5) Since w_1 is a caloric function vanishing on the Lipschitz boundary $\partial\Omega_1 \cap \{t_{i_0} < t < t_{i_0+2}\}$, w_1 is almost harmonic near $\partial\Omega_1 \cap \{t_{i_0+1} \leq t \leq t_{i_0+2}\}$ by Lemma 2.1. Hence (7.4) implies that for $(x,t) \in \Sigma' \cap \{t_{i_0+1} \leq t \leq t_{i_0+2}\}$

$$c(n,M)\text{dist}(x,\partial B_{r(t)}(0)) \leq w_1(x,t) \leq C(n,M)\text{dist}(x,\partial B_{r(t)}(0)). \tag{7.5}$$

(6) Applying Lemma 2.3 to the re-scaled $w_1(\sqrt{t_{i_0+2}}x, t_{i_0+2}t)$, we obtain that for $t \in (t_{i_0+1}, t_{i_0+2})$ and $x \in B_{r(t)}(0) - B_{(1-\alpha^\epsilon)r(t)}(0)$

$$c(n,M) \leq |\nabla w_1(x,t)| \leq C(n,M) \tag{7.6}$$

and

$$|\partial w_1/\partial t| \leq \frac{C(n,M)r(t_{i_0+1})}{t_{i_0+2}} \leq \frac{C(n,M)}{r(t_{i_0+1})} \tag{7.7}$$

where (7.6) and the first inequality of (7.7) follow from (7.5) and the second inequality of (7.7) follows from Lemma 3.1.

(7) From the construction of $\tilde{\psi}_4$,

$$\max \tilde{\psi}_4 = \max_{\partial\Omega_1 \cap \{t_{i_0} \leq t \leq t_{i_0+2}\}} u \leq C(n,M)K\alpha r(t_{i_0})$$

where the last inequality follows from $|\nabla u| \leq C_0 M$ (Lemma 2.5) and the condition (b). Hence by Lemma 2.4,

$$\max_{\partial B_{(1-\alpha^\epsilon)r(t)}(0)} |\nabla \tilde{\psi}_4(x,t)| \leq C(n,M)K\alpha^{1-\epsilon}$$

and

$$\max_{\partial B_{(1-\alpha^\epsilon)r(t)}(0)} |\frac{\partial \tilde{\psi}_4}{\partial t}(x,t)| \leq \frac{C(n,M)K\alpha^{1-2\epsilon}}{r(t)}.$$

Recall that the caloric function w_2 vanishes on the Lipschitz boundary $\partial\Omega_1$ and on $\{t = t_{i_0}\}$, and it has nonzero boundary values $\tilde{\psi}_4$ only on the inner boundary of Σ', i.e., on $\partial B_{(1-\alpha^\epsilon)r(t)}(0) \times \{t\}$. Since the inner boundary of Σ' is also Lipschitz in a parabolic scaling, the above bounds on $|\nabla \tilde{\psi}_4|$ and $|\partial \tilde{\psi}_4/\partial t|$ yield that

$$|\nabla w_2| \leq C(n,M)K\alpha^{1-\epsilon}, \;\; |\partial w_2/\partial t| \leq \frac{C(n,M)K\alpha^{1-2\epsilon}}{r(t)} \text{ in } \Sigma'. \tag{7.8}$$

(8) Combining (7.5), (7.6), (7.7) and (7.8), we obtain

$$c(n,M)\mathrm{dist}(x, \partial B_{r(t)}(0)) \leq w(x,t) \leq C(n,M)\mathrm{dist}(x, \partial B_{r(t)}(0)) \tag{7.9}$$

$$c(n,M) \leq |\nabla w(x,t)| \leq C(n,M) \tag{7.10}$$

and

$$|\partial w/\partial t| \leq \frac{C(n,M)}{r(t_{i_0+1})}. \tag{7.11}$$

(Here recall that $\epsilon = 2/3$ and $\alpha < \alpha(n,M)$ for a sufficiently small $\alpha(n,M)$.)

Now we prove the supercaloricity of v in Π.

$$\begin{aligned}
\Delta v - v_t &\leq 2f\nabla g\cdot\nabla w + gf^2\Delta w - g_t w + g|\nabla w||f'||x| - gw_t\\
&\leq 2f\nabla g\cdot\nabla w - g_t w + g|\nabla w||f'||x| + 2C_1K\alpha g|w_t|\\
&\leq -C(n,M)|\nabla g||\nabla w| + |g_t|w + g|\nabla w||f'||x| + 2C_1K\alpha g|w_t|\\
&\leq \frac{-C(n,M)C_1K\alpha}{\alpha^e r(t_{i_0+1})} + \frac{C'(n,M)C_1K\alpha}{r(t_{i_0+1})} \leq 0
\end{aligned}$$

where the first and second inequalities follow from the construction of v, the third inequality follows from the monotonicity of w_1, i.e., from Lemma 2.2 applied for w_1 with the gradient bounds (7.6) and (7.8) of w_1 and w_2 (note g is radial and increasing in $|x|$), the forth inequality follows from (7.1), (7.2), (7.3), (7.9), (7.10) and (7.11) for constants $C(n,M)$ and $C'(n,M)$ depending on n and M, and the last inequality follows if $\alpha < \alpha(n,M)$ for a sufficiently small $\alpha(n,M) > 0$.

2. For $x \in \partial B_{r(t)/f(t)}(0)$ and $t \in [t_{i_0+1}, t_{i_0+2}]$,

$$\begin{aligned}
|\nabla v(x,t)| &= |w(f(t)x,t)\nabla g(x,t) + g(x,t)f(t)\nabla w(f(t)x,t)|\\
&= |g(x,t)f(t)\nabla w(f(t)x,t)|\\
&\leq (1-2C_1\beta\alpha)|\nabla w| \leq 1
\end{aligned}$$

where the second equality follows since $w = w_1 + w_2 = 0$ on $\partial B_{r(t)}(0)$, the first inequality follows since $f \leq 1 - 2C_1\beta\alpha$ and $g = 1$ on $\partial B_{r(t)/f(t)}(0)$, and the last inequality follows since $w \leq u$ and $\partial\Omega_1$ and $\partial\Omega(u)$ intersect on each small time interval. Hence v is a supersolution of (P) in Π.

3. We show $u \leq v$ on $\Pi \cap \{t = t_{i_0+1}\}$. Recall that $w_1 = \phi$ for $t_{i_0} \leq t < t_{i_0+1}$ and w_1 is not necessarily equal to ϕ at time $t = t_{i_0+1}$ since $\phi(\cdot, t_{i_0+1})$ is defined to be the maximal radial function $\leq u(\cdot, t_{i_0+1})$. However by a similar argument as in the proof of Lemma 6.1, we can show that if the assumptions (a), (b) and (c) of Lemma 6.1 hold for $i = i_0$ and $C = \beta$ then

$$u(\cdot, t_{i_0+1}) \leq w_1(\cdot, t_{i_0+1}) + C(n,M)K\alpha r(t_{i_0+1}) \tag{7.12}$$

on $B_{r(t_{i_0+1})}(0) - B_{(1-\alpha^\epsilon)r(t_{i_0+1})}(0)$ where $C(n,M)$ is a constant depending on n and M. To prove (7.12), recall that

$$u = \psi_1 + \psi_2 + \psi_3 + \psi_4$$

in $\Omega_1 \cap \{t_{i_0} \leq t \leq t_{i_0+1}\}$, where ψ_i are the caloric functions constructed in the proof of Lemma 6.1 with $i = i_0$. From the construction of ψ_1, we can observe

$w_1(\cdot, t_{i_0+1}) = \psi_1(\cdot, t_{i_0+1})$. Also on $B_{r(t_{i_0+1})}(0) \times \{t = t_{i_0+1}\}$

$$\begin{aligned} \psi_2 + \psi_3 + \psi_4 &\leq \beta\alpha^{1-\epsilon}\psi_1 + \psi_3 + \psi_4 \\ &\leq 2\beta\alpha^{1-\epsilon}\psi_1 + \psi_4 \\ &\leq 2\beta\alpha^{1-\epsilon}\psi_1 + C(n,M)K\alpha r(t_{i_0+1}) \end{aligned} \tag{7.13}$$

where the first inequality follows from the construction of ψ_1 and ψ_2 and the condition (a) with $i = i_0$ and $C = \beta$, the second inequality follows from (6.10) with $C = \beta$ and with ψ_1 in place of ϕ, and the last inequality follows from (6.12) and Lemma 3.1. Hence on $B_{r(t_{i_0+1})}(0) - B_{(1-\alpha^\epsilon)r(t_{i_0+1})}(0) \times \{t = t_{i_0+1}\}$

$$\begin{aligned} u &= w_1 + \psi_2 + \psi_3 + \psi_4 \\ &\leq (1 + 2\beta\alpha^{1-\epsilon})w_1 + C(n,M)K\alpha r(t_{i_0+1}) \\ &\leq w_1 + C(n,M)(\beta\alpha r(t_{i_0+1}) + K\alpha r(t_{i_0+1})) \\ &\leq w_1 + C(n,M)K\alpha r(t_{i_0+1}) \end{aligned}$$

where the equality follows from $\psi_1 = w_1$, the first inequality follows from (7.13), the second inequality follows from (7.6), and the last inequality follows since $K > 2\beta$. Hence we obtain (7.12).

Now on $\Pi \cap \{t = t_{i_0+1}\}$ $(= B_{r(t_{i_0+1})}(0) - B_{(1-\alpha^\epsilon)r(t_{i_0+1})}(0) \times \{t = t_{i_0+1}\})$

$$\begin{aligned} v(x, t_{i_0+1}) &\geq (1 - C_1K\alpha)w_1((1 - C_1K\alpha)x, t_{i_0+1}) \\ &\geq (1 - C_1K\alpha)(w_1(x, t_{i_0+1}) + c(n,M)C_1K\alpha r(t_{i_0+1})) \\ &\geq w_1(x, t_{i_0+1}) - C(n,M)C_1K\alpha^{1+\epsilon}r(t_{i_0+1}) + C(n,M)C_1K\alpha r(t_{i_0+1}) \\ &\geq w_1(x, t_{i_0+1}) + C(n,M)C_1K\alpha r(t_{i_0+1}) \\ &\geq u(x, t_{i_0+1}) \end{aligned}$$

where the first inequality follows from the construction of v, the second inequality follows from (7.6) with the monotonicity of w_1 (Lemma 2.2), the third inequality follows from (7.6), the forth inequality follows if $\alpha < \alpha(n,M)$ for a sufficiently small $\alpha(n,M) > 0$, and the last inequality follows form (7.12) if we choose a sufficiently large constant C_1 depending on n and M.

4. We show $v \geq u$ on the inner lateral boundary of Π, i.e., on the set $\partial B_{(1-\alpha^\epsilon)r(t)}(0) \times \{t\}$ for $t \in [t_{i_0+1}, t_{i_0+2}]$. By the construction of w,

$$w = w_1 + \tilde{\psi}_4 \text{ on } \partial B_{(1-\alpha^\epsilon)r(t)}(0) \times \{t\}. \tag{7.14}$$

Here recall that $w_1 + \tilde{\psi}_4$ is a caloric function in $\Omega_1 \cap \{t_{i_0} \leq t \leq t_{i_0+2}\}$ with boundary values u on $\partial\Omega_1$, and ϕ on $\{t = t_{i_0}\}$. Then by a similar argument as in (7.13),

$$u - (w_1 + \tilde{\psi}_4) \leq 2\beta\alpha^{1-\epsilon}w_1 \tag{7.15}$$

in $\Omega_1 \cap \{t_{i_0+1} \leq t \leq t_{i_0+2}\}$. Combining (7.14) and (7.15) we obtain that on $\partial B_{(1-\alpha^\epsilon)r(t)}(0) \times \{t\}$ and for $t \in [t_{i_0+1}, t_{i_0+2}]$

$$u \leq w + 2\beta\alpha^{1-\epsilon} w_1 \leq (1 + 2\beta\alpha^{1-\epsilon})w. \tag{7.16}$$

Now for $x \in \partial B_{(1-\alpha^\epsilon)r(t)}(0)$ and $t \in [t_{i_0+1}, t_{i_0+2}]$,

$$\begin{aligned}
v(x,t) &\geq (1 - C_1 K\alpha)w((1 - 2C_1\beta\alpha)x, t) \\
&\geq (1 - C_1 K\alpha)(w(x,t) + C(n,M)C_1\beta\alpha r(t)) \\
&\geq w(x,t) + C(n,M)C_1\beta\alpha r(t) - C(n,M)C_1 K\alpha^{1+\epsilon} r(t) \\
&\geq w(x,t) + C(n,M)C_1\beta\alpha r(t) - C(n,M)C_1 \alpha^{\frac{\epsilon-1}{2}}\beta\alpha^{1+\epsilon} r(t) \\
&\geq w(x,t) + C(n,M)C_1\beta\alpha r(t) \\
&\geq (1 + 2\beta\alpha^{1-\epsilon})w(x,t) \\
&\geq u(x,t)
\end{aligned}$$

where the first inequality follows since w is decreasing in $|x|$, the second and third inequalities follow from (7.10), the fourth inequality follows from the assumption (6.1) with $C = \beta$, that is $K < \alpha^{(\epsilon-1)/2}\beta$, the fifth inequality follows since $\epsilon = 2/3$, the sixth inequality follows from (7.10) if $C_1 = C_1(n, M)$ is chosen sufficiently large, and the last inequality follows from (7.16).

5. Conclude from 1, 2, 3 and 4 that v is a supersolution of (P) in Π such that $v \geq u$ on $\partial\Pi \cap \{t = t_{i_0+1}\}$ and on the inner lateral boundary of Π. By comparison, $v \geq u$ in Π. Recall that $w \leq u$ in Σ'. Hence the free boundary $\Gamma_t(u)$ of u is trapped between $\Gamma_t(v)$ and $\Gamma_t(w)$ for $t_{i_0+1} \leq t \leq t_{i_0+2}$. Now let $d(t)$ be the distance between $\Gamma_t(v)$ and $\Gamma_t(w)$. Then by the construction of v,

$$d(t) = r(t)(\frac{1}{f(t)} - 1).$$

Since $0 < f(t) < 1$ increases in time on the time interval $[t_{i_0+1}, t_{i_0+2}]$, the function $1/f(t) - 1$ decreases in time on $[t_{i_0+1}, t_{i_0+2}]$. Hence we can obtain an improved estimate on the location of the free boundary at the later time $t = t_{i_0+2}$. Since $f(t_{i_0+2}) = 1 - 2C_1\beta\alpha$,

$$d(t_{i_0+2}) \leq 3C_1\beta\alpha r(t_{i_0+2}).$$

We conclude that the condition (b) holds with K replaced by $3C_1\beta$ for a constant C_1 depending on n and M.

6. Lastly, if (b) holds for $K = 3C_1\beta$ and $i \geq i_0 + 2$, i.e.,

$$\Gamma_t(u) \subset B_{(1+3C_1\beta\alpha)r(t)}(0) - B_{r(t)}(0)$$

for $t \geq t_{i_0+2}$, then since $|\nabla u| \leq C_0 M$,

$$u - \phi = u \leq C(n, M)\beta\alpha r(t)$$

on $\partial\Omega_1 \cap \{t \geq t_{i_0+2}\}$. $\square$

Remark 5. Note that in the proof of Lemma 7.1, we use the condition $K \geq 1$ only for (7.13), i.e., for (6.12).

Corollary 7.2. *For $i \geq 2$, the conditions (a), (b) and (c) of Lemma 6.1 hold with C replaced by $h^i C$ for constants $0 < h < 1$ and $C > 0$ depending on n and M. In other words,*

$$u(\cdot, t_i) \leq (1 + Ch^i \alpha^{1-\epsilon})\phi(\cdot, t_i) \text{ on } B_{(1-\alpha^\epsilon)r(t_i)}(0), \tag{7.17}$$

$\Gamma_t(u)$ is contained in the

$$Ch^i \alpha r(t)\text{- neighborhood of } \partial B_{r(t)}(0) \tag{7.18}$$

for $t \in [t_i, t_{i+1}]$, and

$$u(\cdot, t_i) \leq \phi(\cdot, t_i) + Ch^i \alpha^{\frac{\epsilon+1}{2}} r(t_i) \tag{7.19}$$

on $B_{r(t_i)}(0) - B_{(1-\alpha^\epsilon)r(t_i)}(0)$.

Proof. As in the proof of Corollary 6.2, the conditions (a), (b) and (c) are satisfies with constant $C = K = C(n, M) \geq 1$ for $i \geq 2$. Let m be the integer as in Corollary 6.2 and let $\epsilon = 2/3$. Then for $t \geq t_{m+4}$, the constants C and K can be replaced, respectively, by $\beta = \alpha^{\frac{1-\epsilon}{2}} C$ and $C_1 \beta$ (see Corollary 6.2 and Lemma 7.1). Here C_1 is a constant depending on n and M. Then by the condition (b) with the improved constants, for $t \geq t_{m+4}$,

$$\Gamma_t(u) \subset B_{(1+C_1\beta\alpha)r(t)}(0) - B_{r(t)}(0). \tag{7.20}$$

for a constant $C_1 > 0$ depending on n and M. Fix $i \geq m+4$. Decompose $[t_i, t_{i+1}]$ into subintervals of length $\beta\alpha r^{in}(t_i)^2$ and let τ, $\tilde{\tau}$ and Σ be given similarly as in Lemma 3.2, so that $\Sigma = B_{r^{in}(\tau)}(0) \times [\tilde{\tau}, \tau]$, $\tau - \tilde{\tau} = \beta\alpha r^{in}(t_i)^2$ and $V_{[\tilde{\tau},\tau]} \leq C(n, M)/r^{in}(t_i)$. Recall that $V_{[\tilde{\tau},\tau]}$ is the average velocity of $\partial\Omega^{in}$ on $[\tilde{\tau}, \tau]$. Then using the upper bound on $V_{[\tilde{\tau},\tau]}$,

$$r^{in}(t) - r^{in}(\tau) \leq r^{in}(\tilde{\tau}) - r^{in}(\tau) \leq C(n, M)\beta\alpha r^{in}(t_i). \tag{7.21}$$

for all $t \in [\tilde{\tau}, \tau]$. By (7.20) and (7.21) with $|\nabla u| \leq C_0 M$,

$$\max_{\partial B_{r^{in}(\tau)}(0)} u(\cdot, t) \leq C(n, M)\beta\alpha r^{in}(t_i)$$

for all $t \in [\tilde{\tau}, \tau]$. In other words,

$$u \leq C(n, M)\beta\alpha r^{in}(t_i) \text{ on } \partial\Sigma \tag{7.22}$$

Let ψ be a caloric function in Σ constructed as in Lemma 3.2, then by (7.22) and the improved condition (c),

$$\psi(\cdot, \tau) \geq u(\cdot, \tau) - C(n, M)\beta\alpha r^{in}(t_i) \geq (1 - C(n, M)\sqrt{\beta\alpha})u(\cdot, \tau) \tag{7.23}$$

on $\partial B_{(1-c\sqrt{\beta\alpha})r^{in}(t_i)}(0)$. Using (7.23) instead of (3.24), the construction of Ω_1 can be improved so that $\partial\Omega_1 \cap \{t_i \leq t \leq t_{i+1}\}$ is located in the $C(n, M)\beta\alpha r^{in}(t_i)$-neighborhood of $\partial\Omega^{in}$, for $i \geq m+4$. Then using the bound $|\nabla u| \leq C_0 M$ again,

$$\max_{\partial B_{r(t)}(0)} u(\cdot, t) \leq C(n, M)\beta\alpha r(t).$$

Note that the above inequality gives (6.12), (6.15) and (7.13) for $K = C_1\beta < 1$ and $t \geq t_{m+4}$. Then as mentioned in Remarks 4, we iterate Lemma 6.1 and Lemma 7.1 for $K < 1$, improving the approximating region Ω_1 at later times. □

8 Asymptotic behavior of u; Regularity of $\Gamma(u)$

(7.18) of Corollary 7.2 says that the free boundary of u is asymptotically spherical. Using this result, we approximate u by radially symmetric functions w_k supported on $\Omega_1 \cap \{t_k \leq t < T\}$ such that w_k is caloric and its gradient is close to 1 on $\partial\Omega_1$. Then u turns out to be asymptotically self-similar by Lemma 2.8, and we also obtain the regularity of $\Gamma(u)$ by Lemma 2.9.

Proposition 8.1. *Let $0 < h = h(n, M) < 1$ be as in Lemma 6.1 and let $\epsilon = 2/3$. Then for $k \geq 2$, there exists a radially symmetric caloric function w_k defined in $\Omega_1 \cap \{t_k \leq t < T\}$ such that*

(i) For $t \geq t_k$, $\Gamma_t(u)$ is located in the $Ch^k\alpha r(t)$-neighborhood of $\Gamma_t(w_k)$

(ii) For $t \geq t_k$

$$w_k(\cdot, t) \leq u(\cdot, t) \leq w_k(\cdot, t) + Ch^k\alpha^{1-\epsilon} \max u(\cdot, t) \tag{8.1}$$

where we let $w_k = 0$ outside $\Omega(w_k)$

(iii) On $\partial\Omega(w_k)$

$$1 - Ch^{Ak}\alpha^{A\epsilon} \leq |\nabla w_k| \leq 1 \tag{8.2}$$

where $A = A(n, M) > 0$ is a constant depending on n and M.

In (i), (ii) and (iii) C denotes a constant depending on n and M. By (i), the free boundary $\Gamma_t(u)$ is asymptotically spherical and by Lemma 2.8 with (i), (ii) and (iii), u is asymptotically self-similar.

Proof. Recall that u is well approximated by a radial function ϕ, which is caloric on each time interval (t_i, t_{i+1}). However ϕ does not solve a heat equation on (t_k, T) since it is discontinuous at each t_i ($\phi(\cdot, t_i)$ is defined to be the maximal radial function $\leq u(\cdot, t_i)$). Hence we construct another radial function $w_k \leq \phi$ which is caloric on (t_k, T). Using Corollary 7.2, we show that the values of w_k are close to the values of u and the gradient of w_k is sufficiently close to 1 on its vanishing boundary.

For $k \geq 2$, let w_k solve

$$\begin{cases} \Delta w_k = \partial w_k/\partial t & \text{in} \quad \Omega_1 \cap \{t > t_{k-1}\} \\ w_k = \phi & \text{on} \quad \{t = t_{k-1}\} \\ w_k = 0 & \text{on} \quad \partial\Omega_1 \cap \{t > t_{k-1}\} \end{cases}$$

and let $\tilde{w}_k$ solve

$$\begin{cases} \Delta \tilde{w}_k = \partial \tilde{w}_k/\partial t & \text{in} \quad \Omega_1 \cap \{t > t_{k-1}\} \\ \tilde{w}_k = u & \text{on} \quad \{t = t_{k-1}\} \\ \tilde{w}_k = 0 & \text{on} \quad \partial\Omega_1 \cap \{t > t_{k-1}\}. \end{cases}$$

Then for $t \geq t_k$,

$$w_k \leq \tilde{w}_k \leq (1 + Ch^k \alpha^{1-\epsilon}) w_k \tag{8.3}$$

where $C = C(n, M) > 0$ and the second inequality follows from (7.17) and (7.19) with $\epsilon = 2/3$. For $i \geq k-1$, let v_i be a caloric function defined in $\Omega_1 \cap \{t > t_{k-1}\}$ with the following boundary condition

$$\begin{cases} v_i = 0 & \text{on} \quad \{t = t_{k-1}\} \\ v_i = u & \text{on} \quad \partial\Omega_1 \cap \{t_i < t < t_{i+1}\} \\ v_i = 0 & \text{on} \quad \partial\Omega_1 \cap \{t_{k-1} < t < t_i \text{ or } t > t_{i+1}\}. \end{cases}$$

Then in $\Omega_1 \cap \{t > t_{k-1}\}$

$$u = \tilde{w}_k + \sum_{i=k-1}^{\infty} v_i. \tag{8.4}$$

Throughout the proof, let C denote a positive constant depending on n and M. Then by (7.18) with $|\nabla u| \leq C_0 M$ (Lemma 2.5),

$$v_i = u \leq Ch^i \alpha r(t_i) \tag{8.5}$$

on $\partial\Omega_1 \cap \{t_i < t < t_{i+1}\}$. Hence in $\Omega_1 \cap \{t \geq t_{i+2}\}$,

$$v_i \leq Ch^i \alpha u. \tag{8.6}$$

Combining (8.3), (8.4), (8.5) and (8.6), we obtain that in $\Omega_1 \cap \{t \geq t_k\}$

$$\begin{aligned} u &\leq (1 + Ch^k \alpha^{1-\epsilon}) w_k + \sum_{i=k-1}^{\infty} Ch^i \alpha u + Ch^k \alpha \max u(\cdot, t) \\ &\leq (1 + Ch^k \alpha^{1-\epsilon}) w_k + Ch^k \alpha \max u(\cdot, t) \\ &\leq w_k + Ch^k \alpha^{1-\epsilon} \max u(\cdot, t). \end{aligned} \tag{8.7}$$

Also for $t \in (t_i, t_{i+1})$, $i \geq k$, and $x \in \Omega_t(u) - \Omega_t(w_k)$,

$$u(x,t) \leq Ch^i \alpha r(t_i) \leq Ch^k \alpha r(t_i) \leq Ch^k \alpha^{1-\epsilon} \max u(\cdot, t)$$

where the first inequality follows from a similar argument as in (8.5). Hence we obtain the second part (ii) of the lemma. Observe that the first part (i) follows from Corollary 7.2 since $\Gamma_t(w_k) = \partial B_{r(t)}(0)$ for $t \geq t_k$.

Next we prove (iii) that $|\nabla w_k|$ is sufficiently close to 1 on $\partial\Omega_1 \cap \{t \geq t_k\}$. Since $w_k \leq u$ and the free boundary $\Gamma_t(w_k)$, that is $\partial B_{r(t)}(0)$, intersects $\Gamma_t(u)$ at each t, we obtain the upper bound

$$|\nabla w_k| \leq 1 \text{ on } \partial\Omega_1 \cap \{t \geq t_k\}.$$

To obtain the lower bound of $|\nabla w_k|$, i.e., for the first inequality of (8.2), we compare w_k with some harmonic functions near the vanishing boundary $\partial B_{r(t)}(0)$. Fix a dyadic interval $(t_i, t_{i+1}] \subset (t_k, T)$. For $t \in (t_i, t_{i+1}]$, let $H_{(t)}(\cdot)$ be the harmonic function defined in $B_{r(t)}(0) - B_{(1-h^{k/2}\alpha^\epsilon) r(t)}(0)$ with the following boundary data

$$\begin{cases} H_{(t)} = 0 & \text{on} \quad \partial B_{r(t)}(0) \\ H_{(t)}(\cdot) = w_k(\cdot, t) & \text{on} \quad \partial B_{(1-h^{k/2}\alpha^\epsilon) r(t)}(0). \end{cases}$$

Then by Lemma 2.1 applied to w_k,

$$H_{(t)}(\cdot) \leq (1 + h^{ak/2} \alpha^{a\epsilon}) w_k(\cdot, t) \tag{8.8}$$

on $B_{r(t)}(0) - B_{(1-h^{k/2}\alpha^\epsilon)r(t)}(0)$ where $a > 0$ is a constant depending on n and M. This implies

$$|\nabla H_{(t)}| \leq (1 + h^{ak/2}\alpha^{a\epsilon})|\nabla w_k| \tag{8.9}$$

on $\partial B_{r(t)}(0)$. Let

$$A = A(n, M) = \min\{a/2, 1/2\} > 0.$$

Then by (8.9) it suffices to prove

$$|\nabla H_{(t)}| \geq 1 - Ch^{Ak}\alpha^{A\epsilon} \tag{8.10}$$

on $\partial B_{r(t)}(0)$ for $t \in (t_i, t_{i+1}] \subset (t_k, T)$.

First we show (8.10) for time t in some subset $\{s_1, ..., s_m\}$ of the interval $(t_i, t_{i+1}]$. Recall

(a-1) By Lemma 3.2, the inner-radius $r(t)$ is Lipschitz on $[t_{i-1}, t_{i+1}]$, i.e.,

$$|r(t) - r(s)| \leq C|t - s|/r(t_i)$$

for $t, s \in [t_{i-1}, t_{i+1}]$

(a-2) By Corollary 7.2, the outer-radius $r^{out}(t)$ satisfies

$$r(t) \leq r^{out}(t) \leq r(t) + Ch^i\alpha r(t_i) \leq r(t) + Ch^k\alpha r(t_i)$$

for $t \in [t_{i-1}, t_{i+1}]$.

Also recall that $r^{out}(t)$ is not necessarily Lipschitz on $[t_{i-1}, t_{i+1}]$. However using the properties (a-1) and (a-2), we can construct a Lipschitz function $R(t)$ on $[t_{i-1}, t_{i+1}]$ such that

(b-1) $|R(t) - R(s)| \leq C|t - s|/r(t_i)$ for $t, s \in [t_{i-1}, t_{i+1}]$

(b-2) $r^{out}(t) \leq R(t) \leq r(t) + Ch^k\alpha r(t_i)$

(b-3) $R(t) = r^{out}(t)$ for t in some subset $\{s_1, ..., s_m\}$ of $[t_i, t_{i+1}]$ such that $t_i = s_0 < s_1 < ... < s_m < s_{m+1} = t_{i+1}$ and

$$s_j - s_{j-1} \leq h^k\alpha r^2(t_i)$$

for $1 \leq j \leq m + 1$.

Now let $\tilde{\Omega}$ be a space time region on the time interval $[t_{i-1}, t_{i+1}]$ such that

$$\tilde{\Omega}_t = B_{R(t)}(0)$$

for $t \in [t_{i-1}, t_{i+1}]$. Let $\tilde{u}$ solve

$$\begin{cases} \Delta \tilde{u} = \tilde{u}_t & \text{in} \quad \tilde{\Omega} \\ \tilde{u} = u & \text{on} \quad \{t = t_{i-1}\} \\ \tilde{u} = 0 & \text{on} \quad \partial\tilde{\Omega} \cap \{t_{i-1} < t < t_{i+1}\}. \end{cases}$$

Then by the construction of $R(t)$,

$$u \leq \tilde{u} \leq u + Ch^k \alpha r(t_i) \tag{8.11}$$

where the first inequality follows from the first inequality of (b-2) and the last inequality follows from the last inequality of (b-2) with $|\nabla u| \leq C_0 M$.

Fix $t \in \{s_1, ..., s_m\}$. Then $\Gamma_t(u)$ intersects $\Gamma_t(\tilde{u})$ since $\Gamma_t(\tilde{u}) = \partial B_{r^{out}(t)}(0)$. Let $x_0 \in \Gamma_t(u) \cap \Gamma_t(\tilde{u})$, then by (8.11)

$$|\nabla \tilde{u}(x_0, t)| \geq 1. \tag{8.12}$$

On the other hand, let $\tilde{H}(\cdot)$ be the harmonic function defined in $B_{r^{out}(t)}(0) - B_{(1-h^{k/2}\alpha^{\epsilon})r(t)}(0)$ with the following boundary data

$$\begin{cases} \tilde{H} = 0 & \text{on} \quad \partial B_{r^{out}(t)}(0) \\ \tilde{H} = m & \text{on} \quad \partial B_{(1-h^{k/2}\alpha^{\epsilon})r(t)}(0) \end{cases}$$

where

$$m := \max\{u(x,t) : x \in \partial B_{(1-h^{k/2}\alpha^{\epsilon})r(t)}(0)\} + Ch^k \alpha r(t_i).$$

Then by Lemma 2.1 applied to $\tilde{u}$ with (8.11),

$$\tilde{H}(\cdot) \geq (1 - h^{ak/2}\alpha^{a\epsilon})\tilde{u}(\cdot, t) \tag{8.13}$$

in $B_{r^{out}(t)}(0) - B_{(1-h^{k/2}\alpha^{\epsilon})r(t)}(0)$ where $a = a(n, M) > 0$. Hence on $\partial B_{r^{out}(t)}(0)$,

$$|\nabla \tilde{H}| \geq (1 - h^{ak/2}\alpha^{a\epsilon})|\nabla \tilde{u}(x_0, t)| \geq 1 - h^{ak/2}\alpha^{a\epsilon} \tag{8.14}$$

where the last inequality follows from (8.12).

Now we compare the harmonic functions $H_{(t)}$ and $\tilde{H}$ by comparing their boundary values $w_k(\cdot,t)$ and m on $\partial B_{(1-h^{k/2}\alpha^\epsilon)r(t)}(0)$. (Recall $t \in \{s_1,...,s_m\}$ is fixed.) For $x \in \partial B_{(1-h^{k/2}\alpha^\epsilon)r(t)}(0)$

$$\begin{aligned} m &\le (1+Ch^k\alpha^{1-\epsilon})w_k(x,t) + Ch^k\alpha r(t_i) \\ &\le (1+Ch^{k/2}\alpha^{1-\epsilon})w_k(x,t) \end{aligned} \tag{8.15}$$

where the first inequality follows from (8.7) with the construction of m and last inequality follows since (8.7) and the almost harmonicity of w_k imply that $w_k \approx h^{k/2}\alpha^\epsilon r(t)$ on $\partial B_{(1-h^{k/2}\alpha^\epsilon)r(t)}(0)$. Then on $\partial B_{r(t)}(0)$

$$\begin{aligned} |\nabla H_{(t)}| &\ge (1-Ch^{k/2}\alpha^{1-\epsilon})\frac{w_k|_{\partial B_{(1-h^{k/2}\alpha^\epsilon)r(t)}(0)}}{m}|\nabla\tilde{H}| \\ &\ge 1-Ch^{Ak}\alpha^{A\epsilon} \end{aligned} \tag{8.16}$$

where the first inequality follows from the constructions of $H_{(t)}$ and $\tilde{H}$ with (b-2) and the last inequality follows from (8.14) and (8.15) with the constants $A=\min\{a/2,1/2\}$ and $\epsilon = 2/3$. Hence we obtain the desired inequality (8.10) for time t in the subset $\{s_1,...,s_m\}$ of $(t_i,t_{i+1}]$.

Next we show (8.10) for $t \in (s_{j-1},s_j)$, $1 \le j \le m$. Since ϕ is decreasing in time on each dyadic time interval and the region Ω_1 is shrinking in time, w_k is also decreasing in time. Hence on $\partial B_{(1-h^{k/2}\alpha^\epsilon)r(s_j)}(0)$

$$w_k(\cdot,t) \ge w_k(\cdot,s_j). \tag{8.17}$$

By (a-1) with $|t-s_j| \le s_j - s_{j-1} \le h^k\alpha r^2(t_i)$,

$$0 \le r(t) - r(s_j) \le Ch^k\alpha r(t). \tag{8.18}$$

Then by (8.17) and (8.18) with the almost harmonicity of w_k,

$$w_k(\cdot,t)|_{\partial B_{(1-h^{k/2}\alpha^\epsilon)r(t)}(0)} \ge (1-Ch^{k/2}\alpha^{1-\epsilon})w_k(\cdot,s_j)|_{\partial B_{(1-h^{k/2}\alpha^\epsilon)r(s_j)}(0)}. \tag{8.19}$$

Hence we obtain

$$\begin{aligned} |\nabla H_{(t)}|_{\partial B_{r(t)}(0)} &\ge (1-Ch^{k/2}\alpha^{1-\epsilon})|\nabla H_{(s_j)}|_{\partial B_{r(s_j)}(0)} \\ &\ge 1-Ch^{Ak}\alpha^{A\epsilon} \end{aligned}$$

where the first inequality follows from the construction of $H_{(t)}$ with (8.18) and (8.19), the last inequality follows from (8.16). Since $t \in (s_{j-1},s_j]$ for $1 \le j \le m$, we can conclude that (8.10) holds for all $t \in (s_0,s_m] = (t_i,s_m] \subset (t_i,t_{i+1}]$. Then by repeating the above argument with t_{i+1} replaced by t_{i+2}, we can obtain (8.10) for all $t \in (t_i, t_{i+1}]$. Recall that (8.10) implies the first inequality of (8.2). Hence we obtain the properties (i), (ii) and (iii) of the proposition for the radial function w_k for $k \geq 2$.

Observe that by (i) and (ii),

$$\sup_{t_k<t<T} \|u(\cdot,t) - w_k(\cdot,t)\|_\infty / \|u(\cdot,t)\|_\infty \to 0 \tag{8.20}$$

and

$$\sup_{t_k<t<T} \operatorname{dist}(\Gamma_t(u), \Gamma_t(w_k))/r(t) \to 0 \tag{8.21}$$

as $k \to \infty$ where $r(t) =$ diameter of $\Gamma_t(w_k)/2$. On the other hand, (iii) implies that the radial function w_k is a supersolution of (P) and also the function $(1 + Ch^{Ak}\alpha^{A\epsilon})w_k$ is a subsolution of (P), both of which vanish at time $t = T$. Hence for some constant $1 \leq \beta \leq 1+Ch^{Ak}\alpha^{A\epsilon}$, a radial solution v of (P) vanishes at time $t = T$ if v has an initial condition $v(\cdot, t_{k-1}) = \beta w_k(\cdot, t_{k-1}) = \beta\phi(\cdot, t_{k-1})$. Then by a similar argument as in the proof of Lemma 3.1, we can show that the free boundary $\Gamma_t(v)$ is located in the $C(n,M)h^{Ak}\alpha^{A\epsilon}r(t)$-neighborhood of $\Gamma_t(w_k)$ since v and w_k, otherwise, would have different extinction times. Then using the upper bounds of $|\nabla w_k|$ and $|\nabla v|$ (Lemma 2.5),

$$|v(\cdot,t) - w_k(\cdot,t)| \leq C(n,M)h^{Ak}\alpha^{A\epsilon}r(t) \tag{8.22}$$

where $r(t) \approx \|w_k(\cdot,t)\|_\infty$. By Lemma 2.8, v is asymptotically self-similar and hence we can conclude from (8.20), (8.21) and (8.22) that u is asymptotically self-similar. □

The next corollary follows from Lemma 2.9 and the flatness of $\Gamma(u)$. Note that it was proved in [W] that a limit solution of (P) is also a solution in the sense of domain variation.

Corollary 8.2. *Let $0 < h = h(n,M) < 1$ be as in Lemma 6.1. Then there exists a constant $c_0 > 0$ depending on n and M such that if $h^k\alpha \leq c_0$ for some $k \geq 2$, then $\Gamma(u) \cap \{t_k < t < T\}$ is a graph of $C^{1+\gamma,\gamma}$ function and the space normal is Hölder continuous.*

Proof. Let σ_1 be the constant as in Lemma 2.9. Let

$$(y,\tau) \in \Gamma(u) \cap \{t_k < t < T\}$$

where k is a sufficiently large integer which will be chosen later. Without loss of generality, we assume that $y = (0,...,0,y_n)$ with $y_n > 0$ and $\tau \in (t_k, t_{k+1}]$. Let

$$\rho = \delta r(\tau)$$

where $\delta > 0$ is a sufficiently small constant depending on n, M and σ_1, which will be chosen later. Let w_k be the caloric function constructed as in the proof of Proposition 8.1. Recall that $\Gamma(w_k) = \partial\Omega_1 \cap \{t_{k-1} < t < T\}$ is Lipschiz in a parabolic scaling (Lemma 3.2). Then by (i) of Proposition 8.1 and the Lipschitz property of $\Gamma(w_k)$,

$$u = 0 \ \text{ in } \ Q^-_\rho(y,\tau) \cap \{x : x_n \geq y_n + \sigma_1 \rho\} \tag{8.23}$$

if $\delta = \delta(n, M, \sigma_1) > 0$ is chosen sufficiently small and $k = k(n, M, h, \alpha, \sigma_1, \delta)$ is chosen sufficiently large so that

$$\frac{C_1 h^k \alpha}{\sigma_1} \leq \delta \leq C_2 \sigma_1 \tag{8.24}$$

where $C_1 = C_1(n, M) > 0$ is sufficiently large and $C_2 = C_2(n, M) > 0$ is sufficiently small.

Next we show

$$|\nabla u| \leq 1 + \sigma_1^3 \ \text{ in } \ Q^-_\rho(y,\tau).$$

Let $\tilde{\Omega} \subset I\!R^n \times [t_{k-1}, t_{k+1}]$ be the Lipschitz region constructed as in the proof of Proposition 8.1, which contains $\Omega(u) \cap \{t_{k-1} \leq t \leq t_{k+1}\}$. Then since $\max\{|\nabla u|^2 - 1, 0\}$ is subcaloric in $\Omega(u) \cap \{t_{k-1} < t < t_{k+1}\}$,

$$|\nabla u|^2 - 1 \leq v \tag{8.25}$$

where v solves

$$\begin{cases} v_t = \Delta v & \text{in} \quad \tilde{\Omega} \\ v = 0 & \text{on} \quad \partial\tilde{\Omega} \cap \{t_{k-1} < t < t_{k+1}\} \\ v = \max\{|\nabla u|^2 - 1, 0\} & \text{on} \quad \{t = t_{k-1}\}. \end{cases}$$

Observe that by Lemma 2.5

$$\max_{\tilde{\Omega}} v \leq (C_0 M)^2.$$

Also by Lemma 2.1, $v(\cdot, t)$ is almost harmonic near its vanishing boundary $\partial\tilde{\Omega}_t$ for $t \in [(t_{k-1} + t_k)/2, t_{k+1}]$. Hence we obtain that for $t \in [(t_{k-1} + t_k)/2, t_{k+1}]$,

$$v(x,t) \leq \sigma_1^3 \ \text{ if } \ \mathrm{dist}(x, \partial\tilde{\Omega}_t) \leq 3\rho = 3\delta r(\tau) \tag{8.26}$$

where $\delta = \delta(n, M, \sigma_1) > 0$ is chosen sufficiently small.

On the other hand, we can observe from the construction of $\tilde{\Omega}$ that its boundary $\partial\tilde{\Omega}_t$ is located in the $Ch^k\alpha r(t_k)$-neighborhood of $\Gamma_t(u)$ for $t \in [t_{k-1}, t_{k+1}]$. Hence for $(x,t) \in \Gamma(u) \cap Q_\rho^-(y,\tau)$,

$$\mathrm{dist}(x, \partial\tilde{\Omega}_t) \le Ch^k\alpha r(t_k) \le \frac{C\delta\sigma_1 r(\tau)}{C_1} \le \delta r(\tau) = \rho \tag{8.27}$$

where the second and the last inequalities follow from (8.24) with a sufficiently large $C_1 = C_1(n, M)$. Then by (8.26) and (8.27),

$$|\nabla u|^2 - 1 \le v \le \sigma_1^3 \quad \text{in} \quad Q_\rho^-(y,\tau) \tag{8.28}$$

where the first inequality follows from (8.25).

By Lemma 2.9 with (8.23) and (8.28), we conclude that $\Gamma(u) \cap \{t_k < t < T\}$ is a graph of $C^{1+\gamma,\gamma}$ function and the space normal is Hölder continuous, where k is an integer satisfying (8.24).

□

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