Homogenization of Neuman boundary data with fully nonlinear operator

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Abstract

We study periodic homogenization problems for second-order nonlinear pde with oscillatory Neumann boundary conditions, in domains with general geometry. Our results are new even for the linear PDEs with non-divergence structure. The key observation in our analysis is the continuity property of the linear approximation of the problem in half-space domains whose normal belongs to “irrational” directions.

1 Introduction

In this paper, we are interested in the homogenization in the second order, uniformly elliptic PDEs of non-divergence form with oscillating Neumann data. To be precise, let $\mathcal{S}^n$ be the normed space of symmetric $n \times n$ matrices and consider the function $F(M) : \mathcal{S}^n \to I\!R$ which satisfies

(F1) F is uniformly elliptic, i.e., there exists constants $0 < \lambda < \Lambda$ such that

$$\lambda\|N\| \leq F(M+N) - F(M) \leq \Lambda\|N\| \text{ for any } N \geq 0;$$

(F2) (homogeneity) $F(tM) = tF(M)$ for any $M \in \mathcal{S}^n$ and $t > 0$. In particular $F(0) = 0$;

(F3) F is continuous with respect to M.

Typical examples of operators which satisfy (F1)-(F3) are the Pucci extremal operators:

$$\mathcal{P}^+(Du(x)) = \lambda \sum_{\mu_i<0} \mu_i + \Lambda \sum_{\mu_i \geq 0} \mu_i; \quad \mathcal{P}^-(Du(x)) = \Lambda \sum_{\mu_i<0} \mu_i + \lambda \sum_{\mu_i \geq 0} \mu_i$$

where $\mu_1, \cdots, \mu_n$ are eigenvalues of $D^2u(x)$.

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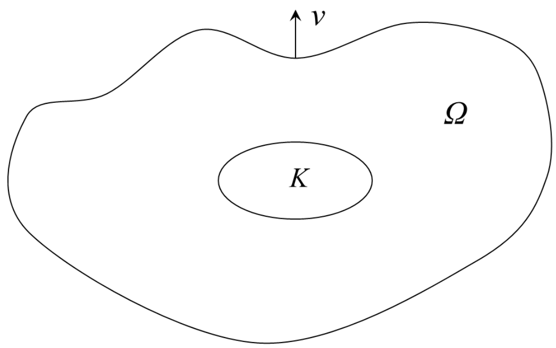

Figure 1

Let $g(x) : I\!R^n \to I\!R$ be a hölder continuous function, which is periodic with respect to the orthonormal basis $\{e_1, ..., e_n\}$ of $I\!R^n$: i.e.,

$$g(x + e_k) = g(x) \text{ for } x \in I\!R^n \text{ and } k = 1, ..., n.$$

Next, let Ω be a domain in $I\!R^n$ which contains a compact set K. (See Figure 1.)

Our goal is to describe the limiting behavior of u_ϵ as $\epsilon \to 0$, where u_ϵ satisfies

$$(P_\epsilon) \qquad \begin{cases} F(D^2 u_\epsilon) = 0 & \text{in} \quad \Omega - K \\ \nu \cdot Du_\epsilon = g(\frac{x}{\epsilon}) & \text{on} \quad \partial\Omega. \\ u = 1 & \text{on} \quad K. \end{cases}$$

Here $\nu = \nu_x$ denotes the outward normal vector at $x \in \partial\Omega$ with respect to Ω. See [7], [8] and [9] for discussion of existence and uniqueness properties of (P_ϵ).

Remark 1.1. *1. Our method can be applied to the operators of the form $F(D^2u, x) = f(x)$ with F and f continuous in x, but we will restrict ourselves to the simple case discussed in (P_ϵ) for the clarity of exposition.*
2. The fixed boundary data on K is introduced to avoid discussion of the compatibility condition on g.
3. The homogeneity condition (F2) can be relaxed (e.g. see (F4) of [2]).

Homogenization of elliptic equations with oscillating coefficients is a classical subject. For the linear, divergence form operator of the form

$$\nabla \cdot (A(\frac{x}{\epsilon})\nabla u_\epsilon) = 0, \tag{1}$$

with the Neumann (co-normal) condition

$$\nu \cdot (A(x/\epsilon)\nabla u)(x) = g(x, x/\epsilon), x \in \partial\Omega, \tag{2}$$

the problem has been widely studied and by now has been well-understood (see [3] for an overview), via the energy method. Here two cases must be distinguished: if $\partial\Omega$ does not contain flat pieces whose normal vectors belong to $I\!R\mathbb{Z}^n$, then u^ϵ converges weakly to the solution of

$$\begin{cases} -\nabla\cdot(A^0\nabla u^0)(x) = 0 & \text{for } x \in \Omega, \\ \nu\cdot(A^0\nabla u^0) = < g > (x) & \text{for } x \in \partial\Omega \end{cases}$$

where $< g > (x) = \int_{[0,1]^n} g(x, \frac{y}{\epsilon})dy$. On the other hand if $\partial\Omega$ does contain a flat piece whose normal vector belongs to $I\!R\mathbb{Z}^n$, then $g(x, x/\epsilon)$ may have a continuum of accumulation points as $\epsilon \to 0$, and thus u^ϵ may have different subsequences converging to different Neumann boundary data. We refer to [3] for details.

On the other hand, for the non-divergence type operator, little is known for the homogenization of the oscillating Neumann boundary data partly due to the lack of energy method. Most available results concern half-space domains with its normal parallel to a vector in $\mathbb{Z}^n$. In [12], Tanaka considered some model problems in half-space whose boundary is parallel to the axes of the periodicity by purely probabilistic methods. In [1], Arisawa studied special cases of problems, again in half-space type domains going through origin, under rather restrictive assumptions, using viscosity solutions as well as stochastic control theory. Generalizing the results of [1], Barles, Da Lio and Souganidis [2] studied the problem for operators with oscillating coefficients, with a series of assumptions which guarantee the existence of approximate corrector.

In this paper we extend above results to the setting of general domains. Before stating the main theorem, let us introduce some notations.

Definition 1.2.

1. *$\nu \in S^{n-1}$ is a rational direction if $\nu \in I\!R\mathbb{Z}^n$.*
2. *$\nu \in S^{n-1}$ is an irrational direction if ν is not a rational direction.*
3. *The domain Ω is irrationally dense if $\partial\Omega$ is C^2 and if $\partial\Omega$ does not contain any flat piece which is normal to a rational vector.*

Now we are ready to state the main results in this paper. We begin with studying half-space type domains.

Theorem 1.3 (Main Theorem I)**.** *Let u_ϵ solve*

$$\begin{cases} F(D^2u_\epsilon) = 0 & in \quad \Sigma := \{x : -1 < (x-p)\cdot\nu < 0\}; \\ \nu\cdot Du = g(\frac{x}{\epsilon}) & on \quad \Gamma_0 := \{x : (x-p)\cdot\nu = 0\}; \\ u = 1 & on \quad \Gamma_I := \{x : (x-p)\cdot\nu = -1\}. \end{cases}$$

Then the following holds:

(i) If ν is irrational, then there is a unique constant $\mu(\nu) \in [\min g, \max g]$ such that u^ϵ locally uniformly converges to the solution of

$$\begin{cases} F(D^2u) = 0 & \text{in} \quad \Sigma \\ \nu \cdot Du = \mu(\nu) & \text{on} \quad \Gamma_0 \\ u = 1 & \text{on} \quad \Gamma_I. \end{cases}$$

(ii) $\mu(\nu) : (S^{n-1} - I\!R\mathbb{Z}^n) \to I\!R$ has a continuous extension $\bar{\mu}(\nu) : S^{n-1} \to I\!R$.

(iii) For rational directions ν, if Γ_0 goes through the origin (that is if $p \cdot \nu = 0$), then the statements in (i) holds for ν as well.

Moreover, we have the following (rough) error estimate: for $\beta = \alpha/4$

$$|u^\epsilon - u| \leq \epsilon^\beta \text{ in } \Omega, \tag{3}$$

where $0 < \alpha < 1$ is the constant given in Theorem 2.3.

Remark 1.4. *1. As shown in [3] for (1) -(2), for a rational direction $\nu \in S^{n-1}$ with $p \cdot \nu \neq 0$, u^ϵ can have different subsequential limits converging to different Neumann data.*

2. The error estimate (3) *is not sharp: we suspect that more careful scaling argument would yield error estimate of up to the order of ϵ^α. Of course, since the estimate is based on the regularity result (Theorem 2.3), further studies on the regularity properties of u_ϵ may produce better estimates. Also see [3](linear case), [5] (nonlinear elliptic PDEs in periodic media) and [6](in random media) for relevant results and discussions on the error estimates.*

Theorem 1.3 (ii) as well as viscosity solutions theory enables the following result in the general domain:

Theorem 1.5 (Main Theorem II)**.** *Let u_ϵ and $\bar{\mu}(\nu)$ be as above and suppose Ω is irrationally dense. Then the solution u_ϵ of (P_ϵ) locally uniformly converges to u solving the following PDE:*

$$\begin{cases} F(D^2u) = 0 & \text{in} \quad \Omega \\ \nu \cdot Du = \bar{\mu}(\nu) & \text{on} \quad \partial\Omega \\ u = 1 & \text{on} \quad K. \end{cases}$$

Remark 1.6 (Open questions)**.** *While our results extend the results of [2] in the case of homogeneous F, the arguments presented here cannot handle the case where the operator F depends on the oscillatory variable $\frac{x}{\epsilon}$: the proof for the continuity property of the limiting slope (Theorem 1.3 (ii)) seems to fail in this case. It remains open to handle oscillation of the operator as well as the oscillation of the boundary data at the same time, in the general domain.*

2 Preliminary results

Let Ω and K be as before, and let $f(x,\nu) : I\!R^n \times S^{n-1} \to I\!R$ be a continuous function. Let us introduce a definition of viscosity solutions for the following problem:

$$(P)_f \qquad \begin{cases} F(D^2u) = 0 & \text{in} \quad \Omega - K; \\ \nu \cdot Du = f(x,\nu) & \text{on} \quad \partial\Omega; \\ u = 1 & \text{on} \quad K \end{cases}$$

The following definition is equivalent to the ones given in [7]:

Definition 2.1. *(a) An upper semi-continuous function $u : \bar{\Omega} \to I\!R$ is a* viscosity subsolution *of $(P)_f$ if u cannot cross from below any C^2 function ϕ which satisfies*

$$F(D^2\phi) > 0 \text{ in } \Omega - K; \quad \nu \cdot D\phi > f(x,\nu) \text{ on } \partial\Omega; \quad \phi \geq 1 \text{ on } K.$$

(b) A lower semi-continuous function $u : \bar{\Omega} \to I\!R$ is a viscosity supersolution *of $(P)_f$ if u cannot cross from above any C^2 function φ which satisfies*

$$F(D^2\varphi) < 0 \text{ in } \Omega - K; \quad \nu \cdot D\varphi > f(x,\nu) \text{ on } \partial\Omega; \quad \varphi \geq 1 \text{ on } K.$$

(c) u is a viscosity solution *of $(P)_f$ if u is both a viscosity sub- and supersolution of $(P)_f$.*

The existence and uniqueness of viscosity solutions of $(P)_f$ follow from the comparison principle we state below:

Theorem 2.2 (Section V, [9])**.** *Let F,K,Ω, ν be as given in the introduction, and let $f : S^{n-1} \to I\!R$ be a continuous function of ν in S^{n-1}. let u and v respectively be sub- and supersolution of $(P)_f$. Then $u \leq v$ in Ω.*

Next we state some regularity results that will be used in the paper.

Theorem 2.3 (Chapter 8, [4]: modified for our setting)**.** *Let u be a viscosity solution of*

$$F(D^2u) = 0$$

in a domain Ω. Then for any $0 < \alpha < 1$ and a compact subset Ω' of Ω, we have

$$\|u\|_{C^\alpha(\Omega')} \leq Cd^{-\alpha}\|u\|_{L^\infty(\Omega)} < \infty$$

for a uniform constant $C > 0$ depending on n, λ, Λ, where $d = d(\Omega', \partial\Omega)$.

Theorem 2.4 (Theorem 8.2, [11]: modified for our setting)**.** *Let*

$$B_1^+ := \{|x| < 1\} \cap \{x \cdot e_n \geq 0\} \text{ and } \Gamma = \{x \cdot e_n = 0\} \cap B_1.$$

Consider $g \in C^\beta(B_1^+)$ for some $\beta \in (0,1)$. Let u solve $F(D^2u) = 0$ in B_1^+ and $\nu \cdot Du = g$ in Γ. Then u is $C^{1,\alpha}(\overline{B_{1/2}^+})$ where $\alpha = \min(\alpha_0, \beta)$ and $\alpha_0 = \alpha_0(n, \lambda, \Lambda)$. Moreover, we have the estimate

$$\|u\|_{C^{1,\alpha}(\overline{B_{1/2}^+})} \leq C(\|u\|_{C(\overline{B_1^+})} + \|g\|_{C^\beta(\Gamma)}),$$

where C is a constant depending only on n, λ, Λ and α.

Weyl's criterion gives us the following Weyl's lemma.

Lemma 2.5 (Weyl's Lemma). *(i) Let α be an irrational number. Then, $k\alpha$ $(mod\, 1)$, for $k \in \mathbb{Z}$, is uniformly distributed on $[0,1]$.*

(ii) Let $\alpha_n = 1$ and $(\alpha_1, \cdots, \alpha_n)$ be an irrational direction. Then,

$$\sum_{i=1}^{n} k_i \alpha_i \ (mod\, 1),$$

for $k_i \in \mathbb{Z}$, is uniformly distributed on $[0,1]$, which means for any subsinterval $A \subset [0,1]$ and $I_N = \{m \in \mathbb{Z}^n : |k_i| < N, i = 1, \cdots, n\}$,

$$\frac{1}{|I_N|} \sum_{k \in I_N} \chi_A(k \cdot \alpha) \to |A|,$$

as $N \to \infty$.

3 In the strip domain

Let us begin with the base case which we will apply to address the general domain.

Fix $p \in \mathbb{R}^n$ and $\nu \in S^{n-1}$ such that $p \cdot \nu \neq 0$. Let

$$\Omega = \Omega_\nu := \{x \in \mathbb{R}^n : -1 \leq (x-p) \cdot \nu \leq 0\} \tag{4}$$

and let u_ϵ be a bounded solution of

$$(PS_\epsilon) \qquad \begin{cases} F(D^2 u_\epsilon) = 0 & \text{in} \quad \Omega \\ \partial u_\epsilon / \partial \nu = g(x/\epsilon) & \text{on} \quad \Gamma_0 := \{x : (x-p) \cdot \nu = 0\} \\ u_\epsilon = 1 & \text{on} \quad \Gamma_I := \{x : (x-p) \cdot \nu = -1\}. \end{cases}$$

Existence and uniqueness for bounded solutions of (PS_ϵ) can be proved via a blow-up process, equi-continuity properties as well as the comparison principle. In fact the following result holds:

Lemma 3.1 (Measurement of side effect). *Suppose $F(D^2 w_1) = F(D^2 w_2) = 0$ in*

$$\Sigma := \Omega \cap \{|x-p| \leq R\}$$

for $R > 2$, with $\partial w_1/\partial \nu = \partial w_2 / \partial \nu$ on Γ_0, $w_1 = w_2$ on Γ_I and $w_1 = 1$, $w_2 = 2$ on $\Omega \cap \{|x-p| = R\}$. Then

$$w_1 \leq w_2 \leq w_1 + \frac{1+3C}{R^2} \ \text{ in } \Omega \cap B_1(p),$$

where $C = \frac{n\Lambda}{\lambda}$.

Proof. Without loss of generality, let us set $\nu = e_n$ and $p = 0$. The first inequality directly follows from the comparison principle. Hence let us show the second inequality.

Let $\tilde{\omega} = w_1 + h_1 + h_2$, where

$$h_1 = \frac{1}{R^2}(|x_1|^2 + ... + |x_n|^2) \text{ and } h_2 = \frac{C}{R^2}(1 - |x_n|^2)$$

for $C = \frac{n\Lambda}{\lambda}$. Note that

$$\begin{aligned} F(D^2\tilde{\omega}) &= F(D^2w_1 + D^2h_1 + D^2h_2) \\ &\leq F(D^2w_1) - \tfrac{2}{R^2}(C\lambda - n\Lambda) \leq F(D^2w_1). \end{aligned}$$

Moreover, $\{x_n = 0\}$, $\tilde{\omega}$ satisfies

$$\partial_{x_n}\tilde{\omega} = \partial_{x_n}\omega_1 = \partial_{x_n}\omega_2.$$

Lastly, on the rest of the boundary of Ω, $\tilde{\omega}$ satisfies $w_2 \leq \tilde{\omega}$. Hence by Theorem 2.2 we have $w_2 \leq \tilde{\omega}$ and we can conclude.

□

In the rest of the paper, we will repeatedly use the fact that linear profiles as well as constants solve $F(D^2u) = 0$.

Lemma 3.2. *Let $0 < \epsilon < 1$. Suppose that w_1 and w_2 solve the equation*

$$(P) \qquad F\left(D^2w_i\right) = 0$$

in Ω with

$$\begin{cases} |w_1 - w_2| \leq \epsilon & \text{on} \quad \Gamma_I \\ \partial w_1/\partial\nu - \partial w_2/\partial\nu = A & \text{on} \quad \Gamma_0. \end{cases}$$

Then in $B_{1/2}(p) \cap \Omega$

$$|w_1 - w_2| \geq C_A - \epsilon$$

where $C_A > 0$ is a constant depending on A.

Proof. Let $\tilde{w} := w_2 + h$, where $h(x) = A(x-p)\cdot\nu + A - \epsilon$. Then $\tilde{\omega}$ satisfies the same Neumann data with ω_1. Further, on Γ_I we have $\tilde{\omega} \leq w_1$. Hence we conclude that $w_2 + h \leq w_1$. Since $h \geq A/2 - \epsilon$ in $B_{1/2}(p)$, we conclude. □

Lemma 3.3. *For $x_0 \in \Omega$, let $H(x_0)$ be the hyperplane, which contains x_0 and is parallel to Γ_0. Let $0 < \epsilon < \mathrm{dist}(x_0, \Gamma_0)$.*

(i) Suppose that ν is a rational direction. Then for any $x \in H(x_0)$, there is $y \in H(x_0)$ such that

$$|x - y| \leq M_\nu\epsilon; \;\; y - x_0 \in \epsilon\mathbb{Z}^n$$

where $M_\nu > 0$ is a constant depending on ν.

(ii) Suppose that ν is an irrational direction. Then for any $x \in H(x_0)$, there is $y \in \Omega$ such that

$$|x-y| \le M_\nu \epsilon^{1/2}; \;\; y - x_0 \in \epsilon \mathbb{Z}^n$$

and

$$\mathrm{dist}(x_0, \Gamma_0) < \mathrm{dist}(y, \Gamma_0) < \mathrm{dist}(x_0, \Gamma_0) + \epsilon^{3/2}$$

where $M_\nu > 0$ is a constant depending on ν.

Proof. (i) follows since for any rational direction ν, there exists an integer $M > 0$ depending on ν such that $M\nu \in \mathbb{Z}^n$. Next, let ν be an irrational direction and let $x \in H(x_0)$. Then by Weyl's Lemma, there exists an integer $M > 0$ depending on ν such that $B_{\epsilon^{-1/2}\epsilon M}(x) \cap H(x_0)$ contains a point z satisfying

$$|x_0 - z| < \epsilon^{1/2}\epsilon, \text{ mod } \epsilon\mathbb{Z}^n.$$

Hence we can find a point y satisfying the conditions in (ii).

□

Lemma 3.4. *Let $\tilde{\Omega} = \Omega + a\nu$ for some $0 \le a \le \epsilon^{3/2}$. Let u_ϵ and $\tilde{u}_\epsilon$ solve (PS_ϵ) in Ω and $\tilde{\Omega}$, respectively. Then in $\Omega \cap \tilde{\Omega}$,*

$$|u_\epsilon - \tilde{u}_\epsilon| \le C\epsilon^{\alpha/2}.$$

Proof. 1. Let $v_\epsilon = \tilde{u}_\epsilon(x + a\nu)$ so that v_ϵ and u_ϵ are defined in the same domain Ω. Since g is continuous, $|\partial v_\epsilon/\partial\nu - \partial u_\epsilon/\partial\nu| \le \epsilon^{1/2}$ on Γ_0.

2. On Γ_I, $u_\epsilon = v_\epsilon = 1$. So now you are talking about two solutions with very small difference in Neumann data. In particular one can compare $u_\epsilon \pm \epsilon^{1/2}(x-p)\cdot\nu$ with $\tilde{u}_\epsilon$ to obtain $|u_\epsilon - v_\epsilon| \le C\epsilon^{1/2}$. Also by Hölder regularity, $|v_\epsilon - \tilde{u}_\epsilon| \le C\epsilon^{\alpha/2}$. Hence we conclude $|u_\epsilon - \tilde{u}_\epsilon| \le C\epsilon^{\alpha/2}$.

□

Lemma 3.5. *Let v and w solve (P) in Ω. If $v = w$ on Γ_I and $v - w \ge A > 0$ on Γ_0, then*

$$v(p - \frac{\nu}{2}) - w(p - \frac{\nu}{2}) \ge C_A > 0.$$

Proof. Due to the boundary condition, $v \ge \omega + A((x-p)\cdot\nu + 1)$. So we conclude. □

The next lemma follows from the $C^{1,\alpha}$ estimates (Lemma 2.4).

Lemma 3.6. *Let v_j be a solution of (PS_ϵ) with a constant Neumann condition $g(x) = \mu_j$. If $\mu_j \to \mu$, then v_j converges to v such that $\partial v/\partial\nu = \mu$ on Γ_0.*

4 Proof of the theorem in a strip region

Here we prove the statements in Theorem 1.3.

4.1 Proof of Theorem 1.3 (i) and (iii)

Suppose $0 \in \Omega = \{-1 \leq (x-p)\cdot\nu \leq 0\}$, which is a point of reference. Recall

$$\Gamma_0 = \{x : (x-p)\cdot\nu = 0\};\ \Gamma_I = \{x : (x-p)\cdot\nu = -1\}.$$

Due to the interior regularity (Theorem 2.3) along subsequences $u_{\epsilon_j} \to u$ uniformly on compact subsets of Ω. Note that there could be different limits along different subsequences ϵ_j, and here we consider one of the subsequential limits.

Denote $u_{\epsilon_j} = u_j$. Then for each j, there exists a constant μ_j and a function v_j in Ω such that

$$\begin{cases} F(D^2 v_j) = 0 & \text{in} \quad \Omega \\ \partial v_j/\partial\nu = \mu_j & \text{on} \quad \Gamma_0 \\ v_j = u_j = 1 & \text{on} \quad \Gamma_I \\ v_j = u_j & \text{at} \quad x = 0. \end{cases}$$

Claim 1. $\mu_j \to \mu$ for some μ as $j \to \infty$. (μ may be different for different subsequences $\{\epsilon_j\}$.)

Proof. Suppose not, then there would be a constant $A > 0$ such that for any $N > 0$, $|\mu_m - \mu_n| \geq A$ for some $m, n > N$. Then by lemma 3.2, $|v_m(0) - v_n(0)| \geq C_A$, which would contradict $v_j(0) = u_j(0) \to u(0)$ as $j \to \infty$. □

Claim 2. Away from the Neumann boundary Γ_0, u_ϵ is almost a constant on hyperplanes parallel to Γ_0. More precisely, let $x_0 \in \Omega$ with $\mathrm{dist}(x_0, \Gamma_0) > \epsilon^{1/4}$. Then there exists a constant $C > 0$ depending on ν, such that for any $x \in H(x_0)$

$$|u_\epsilon(x) - u_\epsilon(x_0)| \leq C\epsilon^{\alpha/4} \tag{5}$$

where $0 < \alpha < 1$.

Proof. First, let ν be a rational direction. Lemma 3.3 implies that for any $x \in H(x_0)$, there is $y \in H(x_0)$ such that $|x - y| \leq M_\nu \epsilon$ and $u_\epsilon(y) = u_\epsilon(x_0)$. Then by Lemma 2.3,

$$|u_\epsilon(x_0) - u_\epsilon(x)| \leq C\epsilon^{-\alpha/4}(M_\nu\epsilon)^\alpha \leq C\epsilon^{\alpha/4}.$$

Next, we assume that ν is an irrational direction and $x \in H(x_0)$. By Lemma 3.3, there exists $y \in \Omega$ such that $|x - y| \leq M_\nu \epsilon^{1/2}$, $y - x_0 \in \epsilon\mathbb{Z}^n$ and

$$\mathrm{dist}(x_0, \Gamma_0) < \mathrm{dist}(y, \Gamma_0) < \mathrm{dist}(x_0, \Gamma_0) + \epsilon^{3/2}.$$

Then we obtain

$$\begin{aligned} |u_\epsilon(x_0) - u_\epsilon(x)| &\le |u_\epsilon(x_0) - u_\epsilon(y)| + |u_\epsilon(y) - u_\epsilon(x)| \\ &\le C\epsilon^{1/2} + |u_\epsilon(y) - u_\epsilon(x)| \\ &\le C\epsilon^{1/2} + C\epsilon^{-\alpha/4}(M_\nu \epsilon^{1/2})^\alpha \\ &\le C\epsilon^{\alpha/4} \end{aligned}$$

where the second inequality follows from Lemma 3.4 and the third inequality follows from Lemma 2.3.

□

By Claim 2 and comparison, we obtain the following estimate: For $x \in \Omega$,

$$|u_\epsilon(x) - v_\epsilon(x)| \le C\epsilon^{\alpha/4} \tag{6}$$

where C is a constant depending on ν.

Claim 3. $\lim v_j = \lim u_j$ *and hence* $\partial u/\partial \nu = \mu$ *on* Γ_0.

Proof. Observe that v_j solves (PS_{ϵ_j}) with $g = \mu_j$. Let x_0 be a point between Γ_0 and $H(0)$. Then by Claim 2, applied to u_j and v_j,

$$|(u_j(x) - v_j(x)) - (u_j(x_0) - v_j(x_0))| \le C\epsilon_j^{\alpha/4}$$

for all $x \in H(x_0)$, if j is sufficiently large. Since $u_j(0) = v_j(0)$, the above inequality and Lemma 3.5 imply that

$$|u_j(x_0) - v_j(x_0)| \to 0 \text{ as } j \to \infty.$$

Hence we get $v_j \to u$ in each compact subset of Ω. By Claim 1 and Lemma 3.6, the limit u of v_j satisfies $\partial u/\partial \nu = \mu$ on Γ_0. □

Claim 4. If ν *is an irrational direction,* $\partial u/\partial \nu = \mu_\nu$ *for a constant* μ_ν *which depends on* ν*, not on the subsequence* ϵ_j*.*

Proof. Let $0 < \eta < \epsilon$ be sufficiently small. After translations, we may let $w_\epsilon(x) := \dfrac{u_\epsilon(\epsilon x)}{\epsilon}$ and $w_\eta(x) := \dfrac{u_\eta(\eta x)}{\eta}$ be defined on the extended strips

$$\Omega_\epsilon := \{x : -\frac{1}{\epsilon} \le (x-p)\cdot\nu \le 0\}$$

and

$$\Omega_\eta := \{x : -\frac{1}{\eta} \le (x-p)\cdot\nu \le 0\}.$$

By Weyl's lemma, we can make translation so that $\partial w_\epsilon/\partial\nu = g(x)$ and $\partial w_\eta/\partial\nu = \tilde{g}(x) := g(x - z_0)$ on Γ_0, where $|z_0| \le \eta$. Observe $|g - \tilde{g}| \le \xi_\eta$ for some $\xi_\eta \to 0$ as $\eta \to 0$.

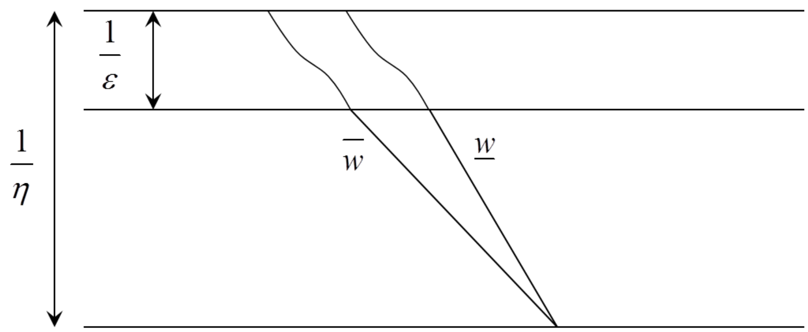

Figure 2

By Claim 2,

$$|w_\epsilon(x) - \frac{v_\epsilon(\epsilon x)}{\epsilon}| \leq \frac{\epsilon^{\alpha/4}}{\epsilon}.$$

Recall that v_ϵ is a solution of the problem (P) with constant Neumann data, which coincides with u_ϵ at the reference point 0 and on Γ_I. Note that v_ϵ is simply a linear profile with slope μ_ϵ.

In particular

$$-h(x) \leq w_\epsilon(x) - \frac{v_\epsilon(\epsilon x)}{\epsilon} \leq h(x) \text{ where } h(x) := \epsilon^{\alpha/4}((x-p)\cdot\nu + 1/\epsilon). \tag{7}$$

2. (7) means that the slope of w_ϵ in the direction of ν (i.e. $\nu \cdot Dw_\epsilon$) is between that of $\mu_\epsilon + \epsilon^{\alpha/4}$ and $\mu_\epsilon - \epsilon^{\alpha/4}$ on $\{x : (x-p)\cdot\nu = -\frac{1}{\epsilon}\}$. Now let us consider linear profiles l_1 and l_2, whose respective slope is $\mu_\epsilon + \epsilon^{\alpha/4}$ and $\mu_\epsilon - \epsilon^{\alpha/4}$, and

$$l_1 = l_2 = \omega_\eta(x) \quad \text{on} \quad \{x : (x-p)\cdot\nu = -\frac{1}{\eta}\}.$$

3. Now we define

$$\bar{w}(x) := \begin{cases} l_1(x) & \text{in} \quad \{-1/\eta \leq (x-p)\cdot\nu \leq -1/\epsilon\} \\ w_\epsilon(x) + c_1 & \text{in} \quad \{-1/\epsilon \leq (x-p)\cdot\nu \leq 0\} \end{cases}$$

and

$$\underline{w}(x) := \begin{cases} l_2(x) & \text{in} \quad \{-1/\eta \leq (x-p)\cdot\nu \leq -1/\epsilon\} \\ w_\epsilon(x) + c_2 & \text{in} \quad \{-1/\epsilon \leq (x-p)\cdot\nu \leq 0\} \end{cases}$$

where c_1 and c_2 are constants satisfying $l_1 = w_\epsilon + c_1$ and $l_2 = w_\epsilon + c_2$ on $\{(x-p)\cdot\nu = -1/\epsilon\}$. (See Figure 2.)

Then due to the observation made in step 2, $\bar{w}$ and $\underline{w}$ are respectively super- and subsolution of (P). Let us define

$$h_1(x) = \eta((x-p)\cdot\nu + 1/\eta).$$

Then $\bar{w} + h_1$ and $\underline{w} - h_1$ are also super- and subsolution of (P). Since $|g - \tilde{g}| < \eta$, by comparison,

$$\underline{w} - h_1 \leq w_\eta \leq \bar{w} + h_1 \quad \text{in } \Omega_\eta. \tag{8}$$

Hence we conclude

$$|\mu_\eta - \mu_\epsilon| \leq \epsilon^{\alpha/4} + \eta,$$

where μ_η is the slope of v_η. □

Claim 4'. If the Neumann boundary Γ_0 passes through $p = 0$, then $\partial u/\partial\nu$ depends on ν, not on the subsequence ϵ_j.

Proof. It can be proved by parallel argument as in Claim 4. Since Ω_ϵ and Ω_η have Neumann boundary passing through 0, $\partial w_\epsilon/\partial\nu = g(x) = \partial w_\eta/\partial\nu$ without translation. □

Remark 4.1. *As mentioned in the introduction, if ν is a rational direction with $p\cdot\nu \neq 0$, the value of $g(\cdot/\epsilon)$ on $\partial\Omega_\epsilon$ and $\partial\Omega_\eta$ may be very different under any translation, and in that case, the proof of Claim 4 fails. In this case u_ϵ may converge to solutions of different Neumann boundary data depending on the subsequences.*

4.2 Proof of Theorem 1.3 (ii)

Proposition 4.2 (Theorem 1.3 (ii))**.** *The homogenized limit $\mu(\nu)$ along irrational directions in S^{n-1} has a continuous extension $\bar{\mu}(\nu)$ over S^{n-1}.*

Proof. Let us fix a unit vector $\nu \in S^{n-1}$. Given $\delta > 0$, we will show that there exists $\epsilon > 0$ depending on the choice of ν such that for any irrational $\nu_1, \nu_2 \in S^{n-1}$,

$$|\mu(\nu_1) - \mu(\nu_2)| < C\delta^{1/2} \text{ whenever } |\nu_1 - \nu|, |\nu_2 - \nu| < \epsilon, \tag{9}$$

where C depends on the choice of ν.

1. For simplicity of proof, we first present the case $n = 2$. Without loss of generality, we may also assume that $\nu = e_n = e_2$ and $p = 0$. We point out that in the proof presented below it does not make any difference in proof if ν were irrational, because here we do not use periodicity of the boundary. Indeed, as we will see, more delicate proofs are required when ν is a rational direction.

Then we have

$$\Omega_0 := \Omega_\nu = \{(x, y) \in I\!R^2 : -1 \leq y \leq 0\}.$$

Let us define $\Omega_k := \Omega_{\nu_k}$ for $k = 1, 2$, and define the family of functions (see Figure 3)

$$g_i(x_1, x_2) := g(x_1, \delta(i-1)), \text{ where } i = 1, ..., m := [\frac{1}{\delta}] + 1.$$

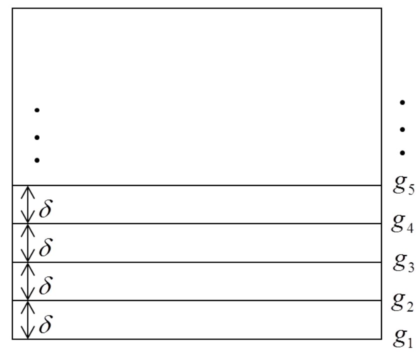

Figure 3

Before moving onto the next step in the proof, Let us briefly discuss the heuristics in the proof.

Proof by heuristics:

Since the domains Ω_1 and Ω_2 point toward with different directions ν_1 and ν_2, we cannot directly compare their boundary data, even if $\partial\Omega_1$ and $\partial\Omega_2$ cover most part of the unit cell in $I\!R^n/Z\!\!\!Z^n$. To overcome this difficulty we perform a two-scale homogenization.

First we consider the functions g_i ($i = 1, .., m$), whose profiles cover most values of g (up to an order of δ). Note that most values of g are taken on $\partial\Omega_k$ for $k = 1, 2$ since ν_k's are irrational. On the other hand, since ν_k's are very close to ν which may be a rational direction, the averaging behavior of a solution u_ϵ would appear only after ϵ gets very small as ν_k approaches ν.

If $|\nu_1 - \nu|$ is chosen much smaller than δ, we can say that the Neumann data $g_1(\cdot/\epsilon)$ is (almost) repeated N times on $\partial\Omega_1$ with period ϵ, up to the error $O(\delta)$. (See Figure 4.) Here N is a sufficiently large number depending on δ and $|\nu_1 - \nu|$. Similarly, on the next piece of boundary $g_2(\cdot/\epsilon)$ is (almost) repeated N times and then $g_3(\cdot/\epsilon)$ is repeated N times: this pattern will repeat with g_k, $k = 3, 4, \ldots$.

Since N is sufficiently large, the solution u_ϵ will exhibit averaged behavior, $N\epsilon$-away from $\partial\Omega_1$. More precisely, on the hyperplane H located $N\epsilon$-away from $\partial\Omega_1$, u_ϵ would be homogenized by the repeating profiles of g_i with an error of $O(\delta)$. This is the first homogenization of u_ϵ near the boundary of Ω_1: we denote the corresponding values of homogenized slopes of u_ϵ on H by $\mu(g_i)$.

Now a unit distance away from $\partial\Omega_1$, we obtain the second homogenization of u_ϵ, whose slope is determined by $\mu(g_i)$, $i = 1, .., m$. Note that this estimation does not depend on the direction ν_1, but on the quantity $|\nu_1 - e_n|$. Hence applying the same argument for ν_2, we conclude that $|\mu(\nu_1) - \mu(\nu_2)|$ is small. $\square$

A rigorous proof of above observation is unfortunately rather lengthy: it is given in step 2.-7. below.

2. Let $\eta := |\nu_1 - e_n|^{8/7}$ and $N = [\frac{\delta}{\eta^{7/8}}]$, define

$$I_1 = [-N\eta, N\eta] \times I\!R$$

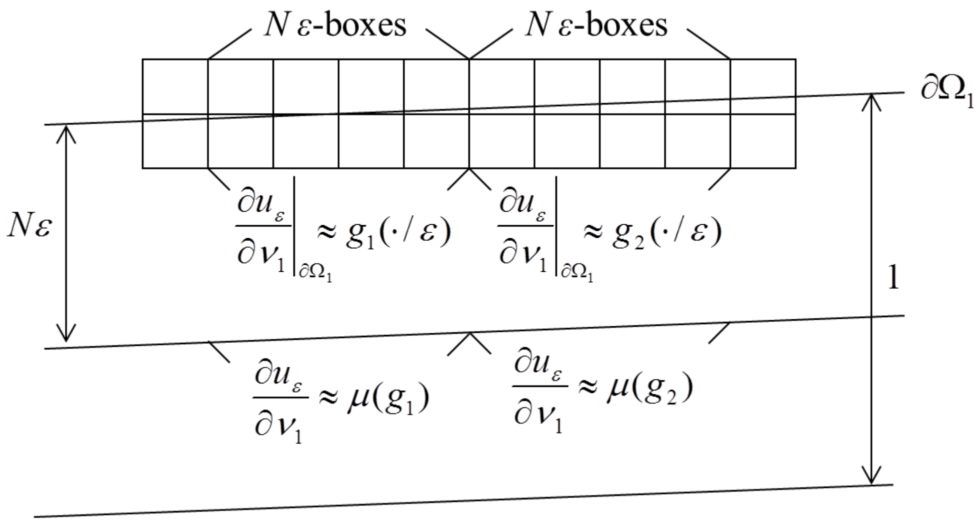

Figure 4

and

$$I_k = [-kN\eta, kN\eta] \times I\!R - \cup_{i=1}^{k-1} I_i, \quad k \in I\!N.$$

Note that g_k is (almost) repeated N times on $I_k \cap \partial\Omega_1$. This and the Lipschitz continuity of g yields that

$$|g(\frac{x}{\eta}, \frac{y}{\eta}) - g_{\tilde{k}}(\frac{x}{\eta})| < \delta \text{ on } \partial\Omega_1 \cap I_k \tag{10}$$

where $\tilde{k} \in [0, m-1]$ denotes k in modulo m.

3. Let w_η solve (P) in Ω_0 with

$$\begin{cases} \frac{\partial w_\eta}{\partial \nu}(x,0) = g_{\tilde{k}}(\frac{x}{\eta}) & \text{for} \quad (x,0) \in I_k \\ w_\eta = 1 & \text{on} \quad \{y=-1\}. \end{cases}$$

Next let u_η solve (P) in Ω_1 with

$$\begin{cases} \frac{\partial u_\eta}{\partial \nu_1}(x,0) = g(\frac{x}{\eta}, \frac{y}{\eta}) & \text{on} \quad \{(x,y)\cdot\nu_1 = 0\}, \\ u_\eta = 1 & \text{on} \quad \{(x,y)\cdot\nu_1 = -1\}. \end{cases}$$

Then by (10)

$$|\mu(w_\eta) - \mu(u_\eta)| < C\delta, \tag{11}$$

where $\mu(w_\eta)$ is the slope of the linear approximation of w_η, as given in Claim 3. Note that $\mu(w_\eta)$ is unique since ν_1 is irrational.

4. Next, we will approximate $\frac{\partial w_\eta}{\partial \nu}$, $N\eta$-away from $\partial\Omega_0$, using its linear approximation v_η which we will define below. Let

$$H := \partial\Omega_0 - N\eta\nu = \partial\Omega_0 - N\eta e_2 = \{(x,y) : y = -N\eta\}.$$

Then for any $x, y \in H$ with $|x-y| < N\eta\delta^{-1/2}$, we can find $z \in H$ such that

$$x = z \text{ modulo } \eta\mathbb{Z}^2 \text{ and } |z-y| \leq \eta.$$

Then as in Claim 2, for α given in Lemma 2.3

$$\begin{aligned} |w_\eta(x) - w_\eta(y)| &\leq |w_\eta(x) - w_\eta(z)| + |w_\eta(z) - w_\eta(y)| \\ &\leq \delta^{1/2} + C\eta^\alpha (N\eta)^{-\alpha} \\ &\leq \delta^{1/2} + C(\frac{\eta^{7/8}}{\delta})^\alpha < C\delta^{1/2} \end{aligned} \tag{12}$$

if η is sufficiently small compared to δ.

On the other hand, there exists a constant μ_η and a linear solution v_η such that

$$\begin{cases} v_\eta = w_\eta(-N\eta\nu) = w_\eta(-N\eta e_2) & \text{on} \quad H = \{y = -N\eta\} \\ \frac{\partial v_\eta}{\partial \nu} = \mu_\eta & \text{on} \quad \partial\Omega_0 = \{y = 0\} \\ v_\eta(-\frac{N\eta\nu}{2}) = w_\eta(-\frac{N\eta\nu}{2}). \end{cases}$$

Now (12) and Lemma 3.1 imply that

$$|v_\eta - w_\eta| \leq C((\frac{1}{N})^{\alpha/4} + \delta^{1/2}) \leq C\delta^{1/2} \text{ in } B_{N\eta\delta^{-1/4}}(0).$$

Let $\mu_{1/N}(g_1)$ be the slope of the linear approximation of a solution whose Neumann data is $1/N$-periodic with profile g_1. By a parallel argument as in (7)

$$\mu_{1/N}(g_1) - C\delta^{1/2} \leq \frac{\partial w_\eta}{\partial \nu} \leq \mu_{1/N}(g_1) + C\delta^{1/2} \text{ on } H \cap I_1. \tag{13}$$

Similarly arguments applies to g_k to yield the following:

$$\mu_{1/N}(g_{\tilde{k}}) - C\delta^{1/2} \leq \frac{\partial w_\eta}{\partial \nu} \leq \mu_{1/N}(g_{\tilde{k}}) + C\delta^{1/2} \text{ on } H \cap I_k. \tag{14}$$

5. Parallel arguments as in step 2. 4. applies to the other direction ν_2: if we define $\bar{\eta}$, $\bar{N}$ and $\bar{H}$ by

$$|\nu_2 - e_2| = \bar{\eta}^{7/8}, \quad \bar{N} = [\frac{\delta}{\bar{\eta}^{7/8}}], \text{ and } \bar{H} = \{y = -\bar{N}\bar{\eta}\},$$

then we have

$$\mu_{\frac{1}{\bar{N}}}(g_k) - C\delta^{1/2} \leq \frac{\partial w_{\bar{\eta}}}{\partial \nu} \leq \mu_{\frac{1}{\bar{N}}}(g_k) + C\delta^{1/2} \text{ on } \bar{H} \cap \bar{I}_k. \tag{15}$$

6. By Claim 4,

$$|\mu_{\frac{1}{N}}(g_k) - \mu_{\frac{1}{\bar N}}(g_k)| < (\frac{1}{N})^{\alpha/4} + \frac{1}{\bar N} < \delta \tag{16}$$

if η and $\bar\eta$ are sufficiently small compared to δ. Let us denote $\mu_{\frac{1}{N}}(g_k) = \mu_{k,N}$ and let us consider h and $\bar h$ solve

$$\begin{cases} -\Delta h = 1 & \text{in} \quad \{-1 \le y \le -N\eta\} \\ h = 1 & \text{on} \quad \{y = -1\} \\ \frac{\partial h}{\partial \nu} = \mu_{\tilde k,N} & \text{on} \quad H \cap I_k \end{cases}$$

and

$$\begin{cases} -\Delta \bar h = 1 & \text{in} \quad \{-1 \le y \le -N\bar\eta\} \\ \bar h = 1 & \text{on} \quad \{y = -1\} \\ \frac{\partial \bar h}{\partial \nu} = \mu_{\tilde k,\bar N} & \text{on} \quad \bar H \cap \bar I_k \end{cases}$$

Let $\mu(h)$ and $\mu(\bar h)$ be the respective linear approximation for h and $\bar h$. Due to (16), it follows that

$$|\mu(h) - \mu(\bar h)| < C\delta \tag{17}$$

Lastly, observe that by (14) and (15),

$$|\mu(w_\eta) - \mu(h)| < C\delta^{1/2} \text{ and } |\mu(w_{\bar\eta}) - \mu(\bar h) < C\delta^{1/2},$$

yielding

$$|\mu(w_\eta) - \mu(w_{\bar\eta})| < C\delta^{1/2}.$$

Then we conclude from (10) that

$$|\mu(u_\eta) - \mu(u_{\bar\eta})| < C\delta^{1/2},$$

proving our claim.

7. For the general dimensions, let us define $g_i : I\!R^{n-1} \to I\!R$ by

$$g_i(x_1, ..., x_{n-1}) = g(x_1, ..., x_{n-1}, \delta(i-1))$$

for $i = 0, 1, ..., m := [\delta^{-1}]$.

Let us also define

$$I_1 := [-N\eta, N\eta]^{n-1} \times I\!R,$$

and for integers $k > 1$

$$I_k := [-kN\eta, KN\eta]^{n-1} \times I\!R - \cup_{i=1}^{k-1} I_i.$$

Then parallel arguments as in step 1.-6. would apply to yield the lemma for $\nu = e_n$ and $p = 0$.

□

5 Proof of main theorems in general domains

In this section we will use the results obtained in the strip domains as well as stability properties of viscosity solutions to derive the main theorems.

First we show that the solution in Ω near a point $p \in \partial\Omega$ can be approximated by corresponding solutions in strip domains. Let Ω be a bounded domain with C^2 boundary. Suppose $p \in \partial\Omega$ and Ω has the irrational normal direction ν at p. Let

$$L_0 := \{x : \nu \cdot (x - p) = 0\} \text{ and } L_1 = L_0 - \epsilon^k \nu,$$

where $0 < k < 1$ is to be determined.

For the domain

$$\Sigma_k := \Omega \cap \{x : -\epsilon^k < \nu \cdot (x - p) < 0\},$$

Let w_ϵ solve

$$\begin{cases} F(Dw_\epsilon^2) = 0 & \text{in} \quad \Sigma_k \\ \partial w_\epsilon / \partial \nu_x = g(\frac{x}{\epsilon}) & \text{on} \quad \partial\Omega \cap \overline{\Sigma}_k \\ w_\epsilon = 1 & \text{on} \quad L_1 \end{cases}$$

where ν_x is normal to $\partial\Omega$ at $x \in \partial\Omega$. Note that Σ_k has width ϵ^k.

Next, let $\tilde{\Sigma}_k$ be the thin region between L_0 and L_1 and let v_ϵ solve

$$\begin{cases} F(Dv_\epsilon^2) = 0 & \text{in} \quad \tilde{\Sigma}_k \\ \partial v_\epsilon / \partial \nu = g(\frac{x}{\epsilon}) & \text{on} \quad L_0 \\ v_\epsilon = 1 & \text{on} \quad L_1 \end{cases}$$

Since Ω is C^2, we may assume that L_0 is contained in the $\epsilon^{4k/3}$-neighborhood of $\partial\Omega$ in $B_{\epsilon^{2k/3}}(p)$. Then for $x \in L_0$ and $y = x + a\nu \in \partial\Omega$, we have

$$|x - y| \leq \epsilon^{4k/3}. \tag{18}$$

Lemma 5.1. *If k is sufficiently close to 1, then there exists $0 < \beta < 1$ such that*

$$|w_\epsilon - v_\epsilon| \leq \epsilon^{k+\beta}$$

in $\Sigma_k \cap B_{\epsilon^{2k/3}}(p)$.

Proof. Let $p = 0$ for convenience. First note that w_ϵ and v_ϵ will oscillate at most of order ϵ^k in their respective domains $\Sigma_k \cap B_{\epsilon^{k/2}}$ and $\tilde{\Sigma}_k \cap B_{\epsilon^{k/2}}$: This can be checked by comparison with linear profiles, because the strip is ϵ^k -close to the domain in $B_{\epsilon^{k}/2}$ and g oscillates with unit size. Let

$$\tilde{w}(x) = w_\epsilon(\epsilon x)/\epsilon \quad \text{and} \quad \tilde{v}(x) = v_\epsilon(\epsilon x)/\epsilon.$$

Then Theorem 2.4 as well as the fact that g is Lipschitz continuous and $\tilde{w}$ and $\tilde{v}$ oscillates up to ϵ^{k-1} yields that

$$\|\tilde{w}\|_{C^{1,\alpha}}, |\tilde{v}\|_{C^{1,\alpha}} \leq C\epsilon^{k-1}$$

in their respective domains $\frac{1}{\epsilon}\Sigma_k$ and $\frac{1}{\epsilon}\tilde{\Sigma}_k$.

Observe that, due to (18), the Neumann boundary of $\frac{1}{\epsilon}\tilde{\Sigma}_k$ is $\epsilon^{5k/4-1}$ close to that of $\frac{1}{\epsilon}\Sigma_k$ in $B_{\epsilon^{5k/8-1}}$. Therefore we conclude that v_ϵ can be extended to satisfy the Neumann boundary data

$$g(\frac{x}{\epsilon}) + O(\epsilon^{k-1+(\frac{4k}{3}-1)\alpha}) \text{ on } \partial\Omega \cap B_{\epsilon^{5k/8}}.$$

Let us choose k sufficiently close to 1 so that $k-1+(\frac{5k}{8}-1)\alpha > \alpha/6$.

Let $\beta = \alpha/6$. Now by comparison principle we have

$$|w_\epsilon(x) - v_\epsilon(x)| \leq \epsilon^\beta(\nu\cdot(x-p)+\epsilon^k) + h(x) \text{ in } \Sigma_k \cap B_{\epsilon^{5k/8}},$$

where $h(x)$ is the parabola $\epsilon^{-k/4}(x - x\cdot\nu - p)^2$ introduced to control the side effects at $\Sigma_k \cap \partial B_{\epsilon^{5k/8}}$. Hence we conclude by evaluating above upper bound in $\Sigma_k \cap B_{\epsilon^{2k/3}}$.

□

We are now ready to show the main proposition. Let us define

$$\limsup{}^* u^\epsilon(x,t) := \lim_{\epsilon\to 0}(\sup\{u^\epsilon(y,s) : y \in \bar{\Omega}, s \geq 0 \text{ and } |x-y|, |t-s| \leq \epsilon\})$$

and

$$\liminf{}_* u^\epsilon(x,t) := \lim_{\epsilon\to 0}(\inf\{u^\epsilon(y,s) : y \in \bar{\Omega}, s \geq 0 \text{ and } |x-y|, |t-s| \leq \epsilon\}).$$

Proposition 5.2. *Let $\bar{\mu}(\nu) : S^{n-1} \to I\!R$ be the continuous extension of $\mu(\nu)$ obtained in Proposition 4.2. Then*

(a) $\bar{u} := \limsup^* u^\epsilon$ *is the viscosity subsolution of* (P)*;*

(b) $\underline{u} := \liminf_* u^\epsilon$ *is the viscosity supersolution of* (P)*.*

Before proving the proposition, let us first prove the main theorem.

Proof of Theorem 1.5. Due to above proposition and Theorem 2.2, we have $\bar{u} \leq \underline{u}$ in Ω. The locally uniform convergence of u_ϵ then follows from the definition of $\bar{u}$ and $\underline{u}$. □

Proof of Proposition 5.2

1. We will only prove (a), since (b) can be proved via parallel arguments.

2. It follows from standard viscosity solution theory that $F(D^2\bar{u}) \leq 0$ in Ω in the viscosity sense. Also due to interior regularity of u_ϵ it is straightforward to show that $\bar{u} \leq 1$ on K. Therefore if $\bar{u}$ fails to be a subsolution of (P), then there exists a smooth function ϕ which touches $\bar{u}$ from above at a boundary point $x_0 \in \partial\Omega$ and satisfies, for some $\delta > 0$,

$$F(D^2\phi)(x_0) > 0 \text{ and } \frac{\partial\phi}{\partial\nu}(x_0) \geq \mu_\nu + 2\delta \tag{19}$$

where $\nu = nu_{x_0}$. Let us decompose ϕ into $\phi = \phi_1 + \phi_2$ where

$$\phi_1(x) = (x - x_0) \cdot (D\phi - \nu(\nu \cdot D\phi))(x_0).$$

Then

$$\nu \cdot D\phi_1(x_0) = 0 \text{ and } D\phi_2(x_0) = \nu(\nu \cdot D\phi_2)(x_0).$$

Observe that since ϕ_1 is a linear function, ϕ_2 still satisfies (19) instead of ϕ. Furthermore, since ϕ_2 is smooth, we may choose ϵ sufficiently small to replace ϕ_2 (with an error $\epsilon^{4k/3}$) by a linear profile φ with normal ν and

$$\nu \cdot D\varphi \geq \mu_\nu + 2\delta - \epsilon^{k/2} \geq \mu_\nu + \delta$$

in $B_{\epsilon^{2k/3}}(x_0)$.

3. *Case I: when ν_{x_0} is a irrational direction*

To illustrate the idea, first assume that x_0 points toward an irrational direction. Let us consider v_ϵ solving

$$\begin{cases} F(D^2 v_\epsilon) = 0 & \text{in} \quad \Sigma := \{x : -\epsilon^k \leq (x - x_0) \cdot \nu \leq 0\} \\ \nu \cdot Dv_\epsilon = g(\frac{x}{\epsilon}) - C_1 \epsilon^{k/2} & \text{on} \quad \Gamma_0 := \{x : (x - x_0) \cdot \nu = 0\} \\ v_\epsilon = \varphi & \text{on} \quad \Gamma_I := \{x : (x - x_0) \cdot \nu = -\epsilon^k\} \end{cases}$$

where C_1 is the C^2 norm of ϕ near x_0. Note that φ is a constant in the inner strip. From the homogenization result on the strip domain pointing towards an irrational direction (see the proof of Claim 4 in section 4) and a re-scaling argument, it follows that for ϵ sufficiently small depending on δ, we have

$$v_\epsilon \leq \phi - \epsilon^k \delta/2 + C_1 \epsilon^{\beta + k} \text{ in } B_{\epsilon^{2k/3}}(x_0). \tag{20}$$

Here $0 < \beta < 1$ is the constant obtained in (8).

Therefore

$$v_\epsilon(x_0) \leq \phi(x_0) - ce^k \delta/2 \tag{21}$$

Next consider w_ϵ: the viscosity solution of

$$(\tilde{P}) \qquad \begin{cases} F(D^2 w_\epsilon) = 0 & \text{in} \quad \Sigma \cap \Omega \\ \nu \cdot Dw_\epsilon = g(\frac{x}{\epsilon}) - C_1 \epsilon^{k/2} & \text{on} \quad \partial\Omega_0 \\ w_\epsilon = \varphi & \text{on} \quad \Gamma_I := \{x : (x - x_0) \cdot \nu = -\epsilon^k\}. \end{cases}$$

Let us define $\tilde{u}_\epsilon := u_\epsilon - \phi_1 - C_1 \epsilon^{4k/3}$.Then $\tilde{u}_\epsilon$ satisfies $F(D^2 \tilde{u}_\epsilon) = F(D^2 u_\epsilon) = 0$ in Ω, $\tilde{u}_\epsilon \leq \phi_2 - C\epsilon^k \leq \varphi$ in $\Omega \cap B_{\epsilon^{k/2}(x_0)}$ and

$$\nu \cdot D\tilde{u}_\epsilon = g(\frac{x}{\epsilon}) - \nu \cdot D\phi_1(x) \leq g(\frac{x}{\epsilon}) + C_1 \epsilon^{k/2} \text{ on } \partial\Omega \cap B_{\epsilon^{k/2}}(x_0)$$

Therefore $\tilde{u}_\epsilon$ is a viscosity supersolution of $(\tilde{P})$, and due to Theorem 2.2 we have

$$w_\epsilon \leq \tilde{u}_\epsilon \text{ in } \Sigma \cap \Omega.$$

Hence it follows that

$$w_\epsilon(x_0) \le \phi(x_0) - C\epsilon^{4k/3}. \tag{22}$$

Now (21) and (22) contradicts Lemma 5.1 if ϵ is sufficiently small.

4. *Case II: General case*

Finally we discuss the general situation. Due to the fact that Ω is irrationally dense, one can find $y_0 \in \partial\Omega$ pointing toward an irrational direction in arbitrarily small vicinity of a boundary point $x_0 \in \partial\Omega$. Below we will to divide $\partial\Omega$ into small neighborhoods of different sizes to argue as above, but with y_0 in place of x_0.

Let us pick a $\delta > 0$ and any given boundary point $y_0 \in \partial\Omega$ whose normal $\nu_{y_0} = p$ is an irrational direction. Then there exists $\epsilon_0(y_0) = \epsilon(\delta, p)$ such that (20) will hold in $B_{\epsilon^{2k/3}}(y_0)$ for $0 < \epsilon < \epsilon_0$ and we will run into a contradiction with smooth ϕ satisfying (19) and touching u_ϵ from above at a point $x_0 \in \frac{1}{2}B_{\epsilon^{2k/3}}(y_0)$.

Since Ω is irrationally dense, the union of $r(y_0) := \frac{1}{2}(\epsilon_0(y_0))^{2k/3}$-neighborhood of y_0 over all $y_0 \in \partial\Omega$ whose normal is irrational covers all of the $\partial\Omega$. Let us call this covering $\mathcal{N}(\delta)$.

Now suppose $\bar{u}$ fails to be a subsolution of (P). Then as before, there exists a smooth function ϕ which touches $\bar{u}$ from above at a boundary point $x_0 \in \partial\Omega$ and satisfies, for some $\delta > 0$,

$$F(D^2\phi) > 0 \text{ and } \frac{\partial\phi}{\partial\nu} \ge \mu_\nu + 2\delta \text{ in } B_{r(y_0)}(y_0), \tag{23}$$

Now due to above discussion, there exists $y_0 \in \partial\Omega$ such that
$x_0 \in B_{r(y_0)}(y_0) \in \mathcal{N}(\delta)$. Now proceeding as in step 2.-3. would yield a contradiction. □

Remark 5.3. *All the argument in the general domain, with little modification, extends to the Neumann boundary data $g(x, x/\epsilon)$ and the elliptic operator $F(D^2u, x)$ with F being continuous with respect to each variable.*

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