# A Martingale Central Limit Theorem

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We present a proof of a martingale central limit theorem (Theorem 2) due to McLeish (1974). Then, an application to Markov chains is given.

**Lemma 1.** For $n \geq 1$, let $U_n, T_n$ be random variables such that

1. $U_n \to a$ in probability.
2. $\{T_n\}$ is uniformly integrable.
3. $\{|T_n U_n|\}$ is uniformly integrable.
4. $E(T_n) \to 1$.

Then $E(T_n U_n) \to a$.

*Proof.* Write $T_n U_n = T_n(U_n - a) + aT_n$. As $E[T_n] \to 1$, we need only show that $E[T_n(U_n - a)] \to 0$ to finish.

Since $\{T_n\}$ is uniformly integrable, we have $T_n(U_n - a) \to 0$ in probability. Also, both $T_n U_n$ and $aT_n$ are uniformly integrable, and so the combination $T_n(U_n - a)$ is uniformly integrable. Hence, $E[T_n(U_n - a)] \to 0$. $\square$

A key observation for the following is the expansion,

$$\exp(ix) = (1 + ix)\exp(-\frac{x^2}{2} + r(x))$$

where $|r(x)| \leq |x|^3$ for real $x$.

**Theorem 1.** Let $\{X_{nj} : 1 \leq j \leq k_n, n \geq 1\}$ be a triangular array of (any) random variables. Let $S_n = \sum_{1 \leq j \leq k_n} X_{nj}$, $T_n = \prod_{1 \leq j \leq k_n}(1 + itX_{nj})$, and $U_n = \exp(-\frac{t^2}{2}\sum_j X_{nj}^2 + \sum_j r(tX_{nj}))$. Suppose that

1. $E(T_n) \to 1$.

2. $\{T_n\}$ is uniformly integrable.

3. $\sum_j X_{nj}^2 \to 1$ in probability.

4. $\max_j |X_{nj}| \to 0$ in probability.

Then $E(\exp(itS_n)) \to \exp(-\frac{t^2}{2})$.

*Proof.* Let $t$ be fixed. From conditions (3) and (4), bound

$$
\begin{aligned}
|\sum_j r(tX_{nj})| &\leq |t|^3 \sum_j |X_{nj}|^3 \\
&\leq |t|^3 \max_j |X_{nj}| \sum_j X_{nj}^2 = o(1).
\end{aligned}
$$

Then,

$$
\begin{aligned}
U_n &= \exp(-\frac{t^2}{2}\sum_j X_{nj}^2 + \sum_j r(tX_{nj})) \\
&= \exp(-\frac{t^2}{2} + o(1)).
\end{aligned}
$$

This verifies condition (1) of Lemma 1 with $a = \exp(-\frac{t^2}{2})$.

Conditions (2) and (4) of Lemma 1 are our present conditions (2) and (1), respectively. Condition (3) of Lemma 1 follows from the fact

$$|T_nU_n| = |\exp itS_n| = |\exp it\sum_j X_{nj}| = 1.$$

Thus $E(\exp itS_n) = E(T_nU_n) \to \exp(-t^2/2)$. $\square$

**Theorem 2.** Let $\{X_{nj}, 1 \leq j \leq k_n, n \geq 1\}$ be a martingale difference array with respect to nested $\sigma$-fields $\{\mathcal{F}_{nj} : 1 \leq j \leq k_n, n \geq 1\}$, $\mathcal{F}_{nj} \subset \mathcal{F}_{nk}$ for $j \leq k$, such that

1. $E(\max_j |X_{nj}|) \to 0$.

2. $\sum_j X_{nj}^2 \to 1$ in probability.

Then $S_n = \sum_j X_{nj} \Rightarrow N(0,1)$ in distribution.

*Proof.* Define $Z_{n1} = X_{n1}$, and $Z_{nj} = X_{nj} I(\sum_{1\le r\le j-1} X_{nr}^2 \le 2)$ for $2 \le j \le k_n$ and $n \ge 1$. Then $\{Z_{nj} : 1 \le j \le k_n, n \ge 1\}$ is also martingale difference array with respect to $\{\mathcal{F}_{nj}\}$ because

$$E(Z_{nj}|\mathcal{F}_{n(j-1)}) = I\Big(\sum_{r\le j-1} X_{nr}^2 \le 2\Big) E(X_{nj}|\mathcal{F}_{n(j-1)}) = 0.$$

Let now $J = \inf\{j : \sum_{1\le r\le j} X_{nr}^2 > 2\} \wedge k_n$. Then,

$$\begin{aligned} P(X_{nr} \ne Z_{nr} \text{ for some } \ r \le k_n) &= P(J \le k_n - 1) \\ &\le P(\sum_{r\le k_n} X_{nr}^2 > 2) \to 0 \qquad (1) \end{aligned}$$

from the third assumption.

It is also to easy that the variables $\{Z_{nj}\}$ satisfy the conditions of the Theorem 2.

We now show that $\{Z_{nj}\}$ satisfies the conditions of Theorem 1. Let $T_n = \prod_{j\le k_n}(1 + itZ_{nj})$. Since $|(1 + itx)|^2 = (1 + t^2x^2) \le \exp(t^2x^2)$, we have

$$\begin{aligned} |T_n| &= \prod_{1\le r\le J-1} (1 + t^2X_{nr}^2)^{1/2}(1 + t^2X_{nJ}^2)^{1/2} \\ &\le \exp((t^2/2) \sum_{1\le r\le J-1} X_{nr}^2)(1 + |t||X_{nJ}|) \\ &\le \exp(t^2)(1 + |t| \max_j |X_{nj}|). \end{aligned}$$

Since $E(\max_j |X_{nj}|) \to 0$, $\{\max_j |X_{nj}|\}$ is uniformly integrable, and therefore $\{T_n\}$ is uniformly integrable. Also, as $\{Z_{nr}\}$ is a martingale difference array, we have by successive conditioning that $E(T_n) = 1$. Hence, conditons (1), (2) and (4) of Theorem 1 for $\{Z_{nj}\}$ are met.

Clearly condition (3) of Theorem 1 also holds for the array $\{Z_{nj}\}$ in view of (1).

Thus all the conditions of Theorem 1 hold, and we conclude $\sum_{r\le k_n} Z_{nr} \to N(0, 1)$. But, by (1), we have then that $\sum X_{nr} \to N(0, 1)$ also. □

For some applications, the following corollary of Theorem 2 is convenient.

**Theorem 3.** Let $\{Z_j : j \ge 1\}$ be a stationary ergodic sequence such that $\sigma^2 = E[Z_1^2] < \infty$, and $E[Z_{n+1}|\mathcal{F}_n] = 0$ where $\mathcal{F}_n = \sigma\{Z_j : j \le n\}$. Then, we have

$$Y_n = \frac{1}{\sqrt{n}}[Z_1 + \cdots Z_n] \Rightarrow \ \mathrm{N}(0, \sigma^2).$$

*Proof.* Let $X_{nj} = Z_j/\sqrt{n}$ and $\mathcal{F}_{nj} = \mathcal{F}_j$ for $1 \leq j \leq n$ and $n \geq 1$. Then, $\{X_{nj}\}$ is a martingale difference array with respect to $\{\mathcal{F}_{nj}\}$.

We now argue that condition (1) of Theorem 2 is satisfied with $Z_{nj} = Z_j/\sqrt{n}$. It is an exercise to show that for a sequence of identically (not necessarily independent) distributed r.v.'s $\{\eta_j\}$, with finite mean, that

$$\lim_{n\to\infty} \frac{1}{n} E\Big[\max_{1\leq j\leq n} |\eta_j|\Big] \;=\; 0.$$

Given this claim, by stationarity of $\{Z_j\}$ and $E[Z_1^2] < \infty$, and taking $\eta_j = Z_j^2$, $E(\max_j |Z_j|/\sqrt{n}) \to 0$ follows. Finally, as ergodicity of the sequence verifies condition (2) of Theorem 2, Theorem 3 follows from Theorem 2.

The exercise is proved as follows: Truncate

$$\begin{aligned} |\eta_j| &= |\eta_j| 1_{[|\eta_j|\leq M]} + |\eta_j| 1_{|\eta_j|>M]} \\ &= A_j + B_j. \end{aligned}$$

Write

$$\max_j |\eta_j| \;\leq\; \max_j A_j + \max_j B_j.$$

Of course, $(1/n)E[\max_j A_j] \to 0$.

Note, using $E[Y] = \int_0^\infty P(Y \geq x)dx$ for nonnegative $Y$,

$$\begin{aligned} E[\max_j B_j] &= \int_0^\infty P(\max_j B_j \geq x)dx \\ &\leq \int_0^\infty P(\cup_{j=1}^n \{B_j \geq x\})dx \\ &\leq \sum_{j=1}^n \int_0^\infty P(B_j \geq x)dx \\ &= n\int_0^\infty P(B_1 \geq x)dx \;=\; nE[|\eta_1|, |\eta_1| > M]. \end{aligned}$$

Then, $\lim_n (1/n)E[\max_j |\eta_j|] \leq E[|\eta_1|, |\eta_1| > M]$ which given the finite mean of $|\eta_1|$ can be made small as $M$ arbitrary. $\square$

We now present an application of Theorem 3 to finite state Markov chains in discrete time.

**Application.** Let $\Sigma$ be a finite state space with $r$ letters, $|\Sigma| = r$. Let $\{X_i : i \geq 1\}$ be an ergodic Markov chain on $\Sigma$ with transition matrix $P$ starting under the stationary measure $\pi$.

Let also $f : \Sigma \to R$ be a mean-zero function with respect to $\pi$, $E_\pi[f] = 0$. Consider now the sum $S_n = \sum_{i=1}^n f(X_i)$.

The aim of this application is to show that $S_n/\sqrt{n}$ converges in distribution to $N(0, \sigma^2)$ for some $\sigma^2 < \infty$ with the help of Theorem 3.

A preliminary lemma will be useful. Let $I_r$ be the $r \times r$ identity matrix. Also note that $f$ can be represented as a vector, $f = \langle f(i) : i \in \Sigma \rangle \in R^r$.

**Lemma 2.** There is a function $u : \Sigma \to R$ such that $f = (I_r - P)u$.

*Proof.* Write

$$R^r = \mathrm{Null}(I - P^*) \oplus \mathrm{Range}(I - P)$$

where $P^*$ is the adjoint of $P$. Then, as $\pi[I - P] = 0$, and $\pi$ is unique, we have

$$\mathrm{Null}(I - P^*) = \{c\pi : c \in R\},$$

a one-parameter space. However, since $E_\pi[f] = 0$ and so $f \perp \pi$, we must have $f \in \mathrm{Range}(I - P)$. $\square$

We now approximate $S_n/\sqrt{n}$ by a martingale. For $n \geq 1$, define

$$M_n = \sum_{i=1}^n [u(X_i) - (Pu)(X_{i-1})] \quad \text{and} \quad \mathcal{F}_n = \sigma\{X_i : 1 \leq i \leq n\}.$$

From the Markov property, the conditional expectation, $E[u(X_i)|\mathcal{F}_{i-1}] = (Pu)(X_{i-1})$. Therefore, $\{M_n\}$ is martingale sequence with respect to $\{\mathcal{F}_n\}$ with stationary ergodic $L^2(\pi)$ differences.

Write

$$\begin{aligned} \frac{1}{\sqrt{n}} \sum_{i=1}^n f(X_i) &= \frac{1}{\sqrt{n}} \Big[ \sum_{i=1}^n u(X_i) - \sum_{i=1}^n (Pu)(X_i) \Big] \\ &= \frac{M_n}{\sqrt{n}} + \frac{(Pu)(X_0) - (Pu)(X_n)}{\sqrt{n}}. \end{aligned}$$

As $u$ is bounded, the error in the martingale approximation vanishes,

$$[(Pu)(X_0) - (Pu)(X_n)]/\sqrt{n} \to 0.$$

We now compute the variance $\sigma^2$:

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} E_\pi[M_n^2] &= E_\pi[(u(X_1) - (Pu)(X_0))^2] \\ &= E_\pi[u^2 - (Pu)^2] \\ &= \sigma^2. \end{aligned}$$

As long as $f$ is non-constant, $u$ is non-constant and $\sigma^2 > 0$. Also, as $u$ is bounded, $\sigma^2 < \infty$.

Hence, by Theorem 3, we have $S_n/\sqrt{n} \Rightarrow N(0, \sigma^2)$. $\square$

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**References.**

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