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CENTRAL LIMIT THEOREMS FOR ADDITIVE FUNCTIONALS OF THE SIMPLE EXCLUSION PROCESS[1]

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Some invariance principles for additive functionals of simple exclusion with finite-range translation-invariant jump rates $p(i, j) = p(j - i)$ in dimensions $d \geq 1$ are established. A previous investigation concentrated on the case of p symmetric. The principal tools to take care of nonreversibility, when p is asymmetric, are invariance principles for associated random variables and a "local balance" estimate on the asymmetric generator of the process.

As a by-product, we provide upper and lower bounds on some transition probabilities for mean-zero asymmetric second-class particles, which are not Markovian, that show they behave like their symmetric Markovian counterparts. Also some estimates with respect to second-class particles with drift are discussed.

In addition, a dichotomy between the occupation time process limits in $d = 1$ and $d \geq 2$ for symmetric exclusion is shown. In the former, the limit is fractional Brownian motion with parameter 3/4, and in the latter, the usual Brownian motion.

1. Introduction and results. Much of the work in the past few years concerning the dynamical evolution of particle configurations in a simple exclusion system makes strong use of certain symmetry or reversibility assumptions. Roughly speaking, when a simple exclusion process is reversible, then a "duality" relation holds. That is, the process preserves the structure of a basis of n-point functions under time evolution. Essentially, seemingly complicated functions may be decomposed into basis components which may be analyzed with sometimes explicit computations.

However, relatively few theorems are available in the asymmetric case when this duality fails. As duality is not a general feature of symmetric systems, techniques of how to avoid duality arguments are of some interest, even with regard to symmetric systems, such as the zero-range process where there is no duality. As will become apparent, however, in this article we trade reliance on "duality" with dependence on the "attractiveness" of the exclusion system which may also be viewed as a specialized feature.

This paper may be considered a companion paper to [14], [15] and [13] where, among other things, central limit theorems for additive functionals of symmetric, mean-zero and $d \geq 3$ simple exclusion with finite-range jump prob-

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abilities are considered. Informally, the question investigated is for which f does $t^{-1/2}\int_0^t f(\eta(s))\,ds$ converge weakly to a normal distribution $N(0,\sigma^2(f))$ where $\eta(t)$ is the configuration at time t? In this article, we examine this question with new ideas for finite-range exclusion processes with nonzero drift in all dimensions $d\geq 1$ as well as clarify and extend the work in [15] and [13] for mean-zero and $d\geq 3$ systems. In contrast to the cases considered in [14], [15] and [13], we note that the Kipnis–Varadhan central limit theorem [5], valid primarily for reversible Markov chains, does not seem to carry over to systems with nonzero drift in $d=1,2$. The method presented here, valid for all $d\geq 1$, relies on an observation of Kipnis [3] that the simple exclusion system and other attractive processes with product (or FKG) invariant measures possess certain association properties which allow modifications of some central limit theorems of Newman and Wright [8], [10] ([9] is a survey) for sequences of stationary L^2 associated random variables to be applied. In addition, several technical variance estimates perhaps of independent interest, based on a "local balance" estimate on the generator of asymmetric exclusion processes, are required. This balance estimate should hold for other conservative particle systems, and so the techniques here would apply to other attractive systems, for instance, zero-range models with increasing rates.

Loosely speaking, the simple exclusion process follows the motion of a collection of random walks on the lattice $\mathbb{Z}^d$ such that jumps to already occupied vertices are suppressed. More precisely, let $\Sigma=\{0,1\}^{\mathbb{Z}^d}$ be the configuration space and let $\eta(t)\in\Sigma$ be the state of the process at time t. It is convenient to represent the configuration in terms of occupation variables $\eta(t)=\{\eta_i(t): i\in\mathbb{Z}^d\}$ where $\eta_i(t)=0$ or 1 according to whether the vertex $i\in\mathbb{Z}^d$ is empty or full at time t. Let $\{p(i,j): i,j\in\mathbb{Z}^d\}$ be the random walk or single particle transition rates. Throughout this article we concentrate on the translation-invariant finite-range case: $p(i,j)=p(0,j-i)=p(j-i)$ and $p(x)=0$ for $|x|>R$ some integer $R<\infty$. In addition, to avoid technicalities, we assume that the symmetrization $(p(i)+p(-i))/2$ is irreducible. In this context, observe that if p is mean-zero, $\sum_i ip(i)=0$, then p is irreducible if and only if its symmetrization is irreducible. This is not necessarily the case if p has drift, $\sum_i ip(i)\neq 0$.

The evolution of the system $\eta(t)$ is Markovian. Let $\{T_t, t\geq 0\}$ denote the process semigroup and let L denote the infinitesimal generator. On test functions ϕ, $(T_t\phi)(\eta)=E_\eta[\phi(\eta(t))]$ and

$$(L\phi)(\eta)=\sum\eta_i(1-\eta_j)(\phi(\eta^{i,j})-\phi(\eta))p(j-i), \tag{1.1}$$

where $\eta^{i,j}$ is the "exchanged" configuration, $(\eta^{i,j})_i=\eta_j$, $(\eta^{i,j})_j=\eta_i$ and $(\eta^{i,j})_k=\eta_k$ for $k\neq i,j$. The transition rate $\eta_i(1-\eta_j)p(j-i)$ for $\eta\to\eta^{i,j}$ represents the exclusion property alluded to above. The construction of the process on bounded continuous functions, through the Hille–Yosida theorem, and extension to L^2 is detailed in I and IV.4 of [7]. Alternatively, the process may be constructed through graphical representation [7], page 383, VIII.2.

The exclusion system is conservative in that random-walk particles are neither destroyed nor created. For such conservative processes, one expects a family of invariant measures indexed according to particle density ρ. Let P_ρ, for $\rho \in [0,1]$, be the infinite Bernoulli product measure over $\mathbb{Z}^d$ with coin-tossing marginal $P_\rho\{\eta_i = 1\} = 1 - P_\rho\{\eta_i - 0\} = \rho$. It is shown in VIII of [7] that $\{P_\rho : \rho \in [0,1]\}$ are invariant for the process $\eta(t)$ and proved in [12] that the P_ρ are also extremal in the convex set of invariant measures.

Let E_ρ denote expectation with respect to P_ρ. We will denote the process measure with initial distribution P_ρ by P_ρ itself.

Throughout the article, we concentrate on a fixed P_ρ, for $\rho \in (0,1)$.

Define $f:\Sigma \to \mathbb{R}$ to be a local function if $f(\eta)$ depends only upon a finite number of the coordinates $\{\eta_i : i \in \mathbb{Z}^d\}$. Also, define the time t variance,

$$\sigma_t^2(f) = E_\rho\left[\int_0^t f(\eta(s))\,ds\right]^2$$

and define, if it exists, the limiting variance

$$\sigma^2(f) = \lim_{t\to\infty} t^{-1}\sigma_t^2(f). \tag{1.2}$$

Also, define relations "$\approx$," "$\sim$" and "$O(\cdot)$" between sequences $a(t) \geq 0$ and $b(t) > 0$ to signify $a(t) \approx b(t)$ when both $0 < \liminf_{t\to\infty} a(t)/b(t)$ and $\limsup_{t\to\infty} a(t)/b(t) < \infty$, $a(t) \sim b(t)$ when $\lim_{t\to\infty} a(t)/b(t)$ exists and $0 < \lim_{t\to\infty} a(t)/b(t) < \infty$ and $a(t) = O(b(t))$ when $\limsup_{t\to\infty} a(t)/b(t) < \infty$.

THEOREM 1.1. *Consider the simple exclusion process with finite-range, translation-invariant jump rates p whose symmetrization is irreducible. Fix the equilibrium P_ρ for $\rho \in (0,1)$ and let f be a local function.*

(i) *Let $\sum ip(i) \neq 0$.*

In $d \geq 3$, the limit $\sigma^2(f) = 2E_\rho[f(-L)^{-1}f]$ exists and $\sigma^2(f) < \infty \Leftrightarrow E_\rho[f] = 0$. Also, $\sigma^2(f) > 0$.

In $d \leq 2$, for coordinatewise increasing f, the limit $\sigma^2(f) = 2E_\rho[f(-L)^{-1}f]$ exists. Also, if f is of the form $f = f_+ - f_-$ where both f_+ and f_- are local coordinatewise increasing and both $\sigma^2(f_+), \sigma^2(f_-) < \infty$ then $\sigma^2(f) = 2E_\rho[f(-L)^{-1}f] < \infty$ exists. In addition, when f is coordinatewise increasing, we have $\sigma^2(f) > 0$.

(ii) *Let $\sum ip(i) = 0$.*

The limit $0 < \sigma^2(f) = 2E_\rho[f(-L)^{-1}f] < \infty$ exists when f satisfies

$$\frac{d^n}{dy^n} E_y[f(\eta)]\big|_{y=\rho} = 0 \tag{1.3}$$

for $n = 0,1,2$ in $d = 1$, $n = 0,1$ in $d = 2$, and $n = 0$ in $d \geq 3$. Otherwise, $\limsup_{t\to\infty} t^{-1}\sigma_t^2(f) = 2E_\rho[f(-L)^{-1}f] = \infty$.

Moreover, when $\sigma^2(f) < \infty$ as in cases (i) and (ii), we have weak convergence in the Skorohod topology on $D[0,\infty)$ to Brownian motion, with respect to the initial measure P_ρ:

$$\lim_{\alpha\to\infty} \alpha^{-1/2}\int_0^{\alpha t} f(\eta(s))ds = B(\sigma^2(f)t).$$

Note, in the case p symmetric, the theorem has already been proved in [14].

When p is mean-zero, $\sum_i ip(i) = 0$, the methods of [15] prove the invariance principle for "H_{-1}" functions f for which it is known that $\sigma^2(f) < \infty$. The contribution here is to identify explicitly this class of functions and to show that if f is not in this class then $\sigma^2(f) = \infty$. In addition, we show the positivity $\sigma^2(f) > 0$ in this mean-zero case.

When p is with drift, $\sum_i ip(i) \neq 0$, and $d \geq 3$, the techniques of [13] prove the invariance principle. The extension here, in these dimensions, is to state the positivity $\sigma^2(f) > 0$, and to provide a different argument which gives stronger convergence to Brownian motion in the uniform topology.

The main contribution of the article is to address the remaining case, when p has non-zero drift and $d \leq 2$. We prove the invariance principle for a class of functions f which have finite limiting variance, $\sigma^2(f) < \infty$. The result is, however, unfinished in some sense, as the class is not exactly identified.

To this point, note that, by the methods of case (i) for $d \geq 3$, it would be possible to write the case (i) results for $d \leq 2$ as simply $\sigma^2(f) < \infty \Leftrightarrow E_\rho[f] = 0$, mimicking the statement for $d \geq 3$, if one could show that $\sigma^2(\eta_0 - \rho) < \infty$ in $d \leq 2$ and $\sigma^2((\eta_0 - \rho)(\eta_1 - \rho)) < \infty$ in $d = 1$. Our conjecture, based on some heuristics (see section 6), is that $\sigma^2(\eta - \rho) < \infty \Leftrightarrow \rho \neq 1/2$ in $d \leq 2$, but that $\sigma^2((\eta_0 - \rho)(\eta_1 - \rho)) < \infty$ for all $0 \leq \rho \leq 1$ in $d = 1$. These curious bounds are open at the moment.

Observe also, in this case, that it may be possible that $\sigma^2(f) = 0$ for some non-monotone function f in $d \leq 2$. It would be of interest to investigate this possibility.

Finally, the quadratic form $E_\rho[f(-L)^{-1}f]$ is understood in the resolvent sense and is discussed in section 3. Also, simple equivalents of the conditions $d^n/dy^n E_y[f]|_{y=\rho} = 0$ are given in subsection 2.2.1.

Now, as a byproduct of the proof, we calculate some divergence rates for $\sigma_t^2(f)$ in the case $\sum ip(i) = 0$:

PROPOSITION 1.1. *Let p be a finite-range, translation-invariant, mean-zero irreducible jump rate. Let also f be a local mean-zero function, $E_\rho[f] = 0$.*

(i) *In $d = 1$, when $d/dyE_y[f]|_{y=\rho} \neq 0$, $\sigma_t^2(f) \approx t^{3/2}$.*
(ii) *In $d = 2$, when $d/dyE_y[f]_{y=\rho} \neq 0$, $\sigma_t^2(f) \approx t\log(t)$.*
(iii) *In $d = 1$, when $d^2/dy^2E_y[f]_{y=\rho} \neq 0$, but $d/dyE_y[f]|_{y=\rho} = 0$,*

$$0 < \liminf_{t\to\infty} \left(t^{-1}\int_0^t \sigma_s^2(f)\,ds\right)/(t\log(t)) \quad \text{and} \quad \limsup_{t\to\infty}(\sigma_t^2(f))/(t\log(t)) < \infty.$$

In fact, when p is symmetric, $\sigma_t^2(f) \approx t\log(t)$.

Note, in case (3) one expects $\sigma_t^2(f) \approx t\log(t)$ for all mean-zero p, not only for symmetric jump rates. However, we could not obtain a better lower bound.

When $\sigma_t(f)$ is superdiffusive, partial results are available. The following occupation time theorem is proved in [4], for finite-range translation-invariant symmetric simple exclusions: let p be jump probabilities with covariance σ_p^2. Then, with respect to initial configurations P_ρ, as $t \uparrow \infty$,

$$\frac{1}{\beta(d,t)} \int_0^t (\eta_0(s) - \rho)\, ds \to N(0, \sigma^2(\rho, d)) \quad \text{and} \tag{1.4}$$
$$\sigma_t^2(\eta_0 - \rho) \sim \beta^2(d,t),$$

where $\beta(1,t) = t^{3/4}$ for $d = 1$, $\beta(2,t) = \sqrt{t \log t}$ for $d = 2$ and $\beta(d,t) = \sqrt{t}$ for $d \geq 3$. The coefficient variances are also computed:

$$\begin{aligned}
\sigma^2(\rho,1) &= \rho(1-\rho)\left(8\Big/\left(3\sqrt{2\pi|\sigma_p^2|}\right)\right) && \text{if } d = 1,\\
\sigma^2(\rho,2) &= \rho(1-\rho)\big(2/(\pi \det(\sigma_p^2))\big) && \text{if } d = 2,\\
\sigma^2(\rho,d) &= \rho(1-\rho)\left(2\int_0^\infty P_{0,0}(t)\, dt\right) && \text{if } d \geq 3,
\end{aligned}$$

where $P_{i,j}(t)$ is the transition probablity of d-dimensional random walk corresponding to p.

We state a modest generalization of Kipnis's theorem.

THEOREM 1.2. *Consider simple exclusion with finite-range, translation-invariant, irreducible symmetric rates p. Then, with respect to initial configurations P_ρ, we have weak convergence in the uniform topology,*

$$\lim_{\alpha\to\infty} \frac{1}{\beta(d,\alpha)} \int_0^{\alpha t} (\eta_0(s) - \rho)\, ds = \begin{cases} B_{3/4}(\sigma^2(\rho,1)t), & \text{when } d = 1,\\ B(\sigma^2(\rho,d)t), & \text{when } d \geq 2,\end{cases}$$

where $B_{3/4}$ and B are fractional Brownian motion (with parameter $H = 3/4$) and standard Brownian motion, respectively.

These results may be generalized to local mean-zero f which do not satisfy (1.3): when $d = 1$ and $d/dyE_y[f]|_{y=\rho} \neq 0$, we have $\alpha^{-3/4}\int_0^{\alpha t} f\, ds \to B_{3/4}(Ct)$. When $d = 2$ and $d/dyE_y[f]|_{y=\rho} \neq 0$, we have $(\alpha\log(\alpha))^{-1/2}\int_0^{\alpha t} f\, ds \to B(Ct)$. In both cases, $C = C(f)$, $0 < C < \infty$, may be calculated. What remains is the case $d = 1$ and $d^n/dy^n E_y[f]|_{y=\rho} = 0$ for $n = 1$ but not for $n = 2$. Without explicit asymptotics of $\sigma_t^2(f)$, we cannot prove the related invariance principle, although the central limit theorem holds in this case: $(\sigma_t^2(f))^{-1/2}\int_0^t f\, ds \to N(0,1)$.

Note that similar results for asymmetric mean-zero systems have not been investigated.

The paper is organized into six sections. In Section 2, we state a central limit theorem for associated random vectors and discuss certain necessary properties of simple exclusion used later. In Section 3, we develop notions of various H_1 and H_{-1} spaces so as to prove some technical variance estimates. In Section 4, we prove some variance bounds for mean-zero asymmetric systems. In Section 5, we prove Theorems 1.1 and 1.2. In Section 6, we obtain some occupation time estimates for second-class particles.

2. Associated random vectors and preliminaries for simple exclusion. We gather here some results on associated or FKG random vectors and the simple exclusion process, which will be useful. Specifically, we investigate the diffusive behavior of associated random systems. For the simple exclusion process, we discuss certain bases for local functions, the duality property, the basic coupling, time reversal and some association properties of the model variables.

2.1. *Associated random vectors.* In this subsection, we connect some facts about the diffusion properties of associated (or FKG) random vectors. See [9] for a survey on associated variables.

DEFINITION 2.1. *Consider an m-dimensional L^2 process with stationary increments,*

$$\{\mathbf{v}(t) = (v_1(t), \dots, v_m(t)), t \geq 0\}.$$

We say $\mathbf{v}$ has weakly positive associated increments if

$$E[\phi(\vec{v}(t+s) - \vec{v}(s))\psi(\vec{v}(s))] \geq E[\phi(\vec{v}(t))]E[\psi(\vec{v}(s))]$$

for all coordinatewise increasing functions ϕ and ψ. When the inequality is reversed, $\vec{v}$ is said to have weakly negative associated increments.

For associated processes, we rephrase some useful central limit theorems of Newman [8] and Newman and Wright [10].

THEOREM 2.1. *Let $\vec{v}(t) = (v_1(t), \dots, v_m(t))$ be an m-dimensional vector process in $C[0,\infty)$ with stationary and weakly positive associated increments such that $E[v_i(t)] = 0$ for all i and t. Assume also for all i and j that*

$$\lim_{t\to\infty} \frac{1}{t} E[v_i(t)v_j(t)] = a_{ij} < \infty. \tag{2.1}$$

Then

$$\frac{1}{\sqrt{\alpha}} \vec{v}(\alpha t)\vec{l} \to B(\vec{l} A \vec{l} t) \tag{2.2}$$

weakly in the uniform topology as $\alpha \to \infty$ where $A = (a_{ij})$ is the covariance matrix, $\vec{l} \in \mathbb{R}^m$ and B is standard Brownian motion.

PROOF. The proof follows from Theorem 3 of Newman [8] and Corollary 6 of Newman and Wright [10]. These results are stated in discrete time but extension to continuous time is straightforward. Theorem 3 of [8] applied to $\alpha^{-1/2}\vec{v}(\alpha t) \cdot \vec{l} \ll \alpha^{-1/2}\sum v_i(\alpha t)|l_i|$ ("$\ll$" is notation from [8]) gives convergence of finite-dimensional distributions in (2.2). Consider the inequality

$$P\left\{\sup_{0\leq s\leq 1} |\vec{v}(s) \cdot \vec{l}| > \gamma\right\} \leq \sum_{i=1}^m P\left\{\sup_{0\leq s\leq 1} |v_i(s)| > \frac{\gamma}{m|l_i|}\right\}.$$

Corollary 6 of [10] applied to the terms on the right above yields maximal inequalities sufficient to prove tightness of these finite-dimensional measures in $C[0,\infty)$. Note that, although a stronger form of associativity is assumed in Newman's Theorem 3, a reading of the proof shows only use of weak associativity. Also, in Corollary 6 of [10], weak associativity implies the demimartingale property assumed there. $\square$

2.2. *Some properties of simple exclusion.* Several useful definitions and necessary facts for simple exclusion systems are presented in five parts.

2.2.1. *Centered and monotone bases.* Let $f(\eta)$ be a local function. A useful observation, following from the fact that f takes finitely many values on a finite set of coordinates, is that f may be rewritten in terms of two different types of elementary functions.

Define $C_I^\rho(\eta)$ and $M_I(\eta)$, for $I = (i_1, \ldots, i_k) \subset \mathbb{Z}^d$ composed of distinct vertices ($i_j \neq i_l$ for $j \neq l$) and $k \geq 1$ as "centered" and "monotone" k-point functions, respectively,

$$C_I^\rho(\eta) = (\eta_{i_1} - \rho)(\eta_{i_2} - \rho)\cdots(\eta_{i_k} - \rho),$$

$$M_I^\rho(\eta) = (\eta_{i_1}\eta_{i_2}\cdots\eta_{i_k}) - \rho^k.$$

Including the constant functions, both $\{1, C_I^\rho(\eta) \colon |I| = k \geq 1\}$ and $\{1, M_I \colon |I| = k \geq 1\}$ form bases under which f may be decomposed.

Let $\langle f, g\rangle_\rho = E_\rho[fg]$ define an inner-product on $L^2(P_\rho)$. Observe that as P_ρ is product measure, the "centered" basis is an orthogonal basis. The "monotone" basis, however, although not orthogonal, consists of coordinate-wise increasing functions.

Explicitly, for some constants c_I and m_I,

$$f(\eta) = E_\rho[f] + \sum c_i(\eta_i - \rho) + \sum_{i\neq j} c_{i,j}(\eta_i - \rho)(\eta_j - \rho) + H_C^\rho(\eta) \tag{2.3}$$

and

$$f(\eta) = E_\rho[f] + \sum m_i(\eta_i - \rho) + \sum_{i\neq j} m_{i,j}(\eta_i\eta_j - \rho^2) + H_M^\rho(\eta), \tag{2.4}$$

where H_C^ρ and H_M^ρ are a finite number of higher order terms.

We now recast the finite variance conditions (1.3) into simple criteria on the constants c_I. Exact computations give for $k \geq 1$ that

$$\frac{d^n}{dy^n} E_y[f(\eta)]|_{y=\rho} = 0, \qquad n = 0, 1, \ldots, k \Leftrightarrow E_\rho[f], \qquad \sum_i c_i, \ldots, \sum_{I:|I|=k} c_I = 0.$$

In particular,

$$\text{conditions (1.3)} \Leftrightarrow \begin{cases} E_\rho[f], \sum_i c_i, \sum_{ij} c_{i,j} = 0, & \text{if } d = 1, \\ E_\rho[f], \sum_i c_i = 0, & \text{if } d = 2, \\ E_\rho[f] = 0, & \text{if } d \geq 3. \end{cases} \tag{2.5}$$

This leads to the following lemma.

LEMMA 2.1. *Let f be a local mean-zero function which violates conditions* (1.3). *Then f may be put in the form*

$$f(\eta) = \left(\sum_i c_i\right) C_0^\rho + \left(\sum_{i,j} c_{i,j}\right) C_{(0,1)}^\rho + \bar{f} \quad \text{in } d = 1 \text{ and}$$

$$f(\eta) = \left(\sum_i c_i\right) C_0^\rho + \bar{f} \quad \text{in } d = 2,$$

where $\sum_i c_i$ and $\sum_{i,j} c_{i,j}$ cannot both vanish in $d = 1$, $\sum_i c_i \neq 0$ in $d = 2$ and $\bar{f}$ satisfies conditions (1.3).

PROOF. As f is local, the decomposition (2.3) is finite. Note also that $E_y[C_I^\rho - C_J^\rho] \equiv 0$ for $|I| = |J|$ and also that $d/dyE_y[C_0^\rho] = 1$ and $d^2/dy^2 E_y[C_{(0,1)}^\rho] = 2$. By adding and subtracting terms of the form $C_i^\rho - C_j^\rho$ and $C_{(i,j)}^\rho - C_{(k,l)}^\rho$ in (2.3), the lemma follows from (2.5). $\square$

2.2.2. *Dual relation for symmetric exclusions.* For $I, J \subset \mathbb{Z}^d$ such that $|I| = |J| = k$, let $P_{I,J}^{SE}(t)$ denote the transition probability of k-particles, performing simple exclusion on $\mathbb{Z}^d$ with jump rate $p(\cdot)$, initially at positions I moving to positions J in time t. Note that when $I = \{i\}$ and $J = \{j\}$ are single sites, $P_{i,j}^{SE}(t) = P_{i,j}(t)$ is no more than the usual random walk transition probability.

The "dual relation" for symmetric simple exclusion processes computes the action of the symmetric semigroup T_t on the k-point functions $C_I^\rho(\eta)$, for $|I| = k$, composed of distinct sites in terms of k-particle transitions (see VIII.1 of [7]),

$$T_t C_J^\rho(\eta) = \sum_{K:|K|=k} P_{J,K}^{SE}(t).$$

In particular, for $|I| = |J| = k$,

$$\begin{aligned} \langle C_I^\rho(\eta), T_t C_J^\rho(\eta)\rangle_\rho &= \sum_{K:|K|=k} P_{J,K}^{SE}(t) \langle C_K^\rho(\eta(0)), C_I^\rho(\eta(0))\rangle_\rho \\ &= (\rho(1-\rho))^k P_{J,I}^{SE}(t). \end{aligned} \tag{2.6}$$

And, for $|I| \neq |J|$, $< C_I^\rho(\eta), T_t C_J^\rho(\eta) >_\rho = 0$.

2.2.3. *The basic coupling.* An essential coupling for the simple exclusion process is the basic coupling ([7], VIII.2). Given two systems, one starting from initial configuration η, the other from η', such that $\eta \leq \eta'$ coordinatewise, we have also that at any later time $t \ge 0$ the ordering is preserved, $\eta(t) \le \eta'(t)$. The coupling generator $\bar{L}$, acting on test functions ϕ, is as follows:

$$\bar{L}\phi(\eta, \eta') = \sum_{i,j} \eta_i p(j-i)(\phi(\eta^{ij}), \eta'^{ij}) - \phi(\eta, \eta')) \tag{2.7}$$

$$+\sum_{i,j}(\eta'_i - \eta_i)p(j-i)(\phi(\eta, \eta'^{ij}) - \phi(\eta, \eta')). \tag{2.8}$$

This coupling may be understood in terms of discrepancies. Let $\xi = \eta' - \eta$ denote the discrepancy configuration. In this context, the process $(\eta(t), \xi(t))$ is composed of "first-class" η-particles and "second-class" ξ-particles. These labels are suggested by the following properties observed from the generator: an η-particle does not see the ξ-particles; however, a ξ-particle must exchange places with an η-particle if an η-particle jumps to its position.

A principal use of the basic coupling in the article is to make the following comparison. Let $f(\eta)$ be an increasing function and suppose $\eta(t)$ and $\eta'(t)$ are exclusion systems starting from initial configurations $\eta \le \eta'$. Then, by the basic coupling, we have $f(\eta(t)) \le f(\eta'(t))$ for all $t \ge 0$.

2.2.4. *Time reversal and* L^*. The notions of time reversal at time s and the time-reversed process $\eta^*(\cdot) = \eta(s - \cdot)$ will be useful. We compute that the time-reversed process $\eta^*(\cdot)$ is generated by the operator L^* which acts on test functions ϕ as follows:

$$L^*\phi = \sum \eta_i(1-\eta_j)(\phi(\eta^{i,j}) - \phi(\eta))p(i-j).$$

Observe that the operators L and L^* are adjoints of each other satisfying the property, for test functions ϕ and ψ, that $E_\rho[\phi(\eta)(L\psi(\eta))] = E_\rho[(L^*\phi(\eta))\psi(\eta)]$.

The adjoint system is just another translation-invariant exclusion system, but with reversed jump rates $p(i, j) = p(i - j)$. Therefore, the Bernoulli product measures P_ρ are all invariant and extremal with respect to L^* as with respect to L. Denote T_t^* and E_η^* as the semigroup and expectation with respect to the $\eta^*(\cdot)$ process.

2.2.5. *Exclusion association properties.* We now show that certain variables with respect to simple exclusion are positively associated.

LEMMA 2.2. *Let $f(\eta), g(\eta): \Sigma \to \mathbb{R}$ be both coordinatewise increasing or decreasing functions. Then, $\langle f(\eta), g(\eta)\rangle_\rho \ge E_\rho[f]E_\rho[g]$. And also, for all $t \ge 0$,*

$$\begin{aligned} E_\rho[f(\eta(0))g(\eta(t))] &= E_\rho[f(\eta(0))E_{\eta(0)}[g(\eta(t))]] \\ &= \langle f(\eta), T_t g(\eta)\rangle_\rho \\ &\ge E_\rho[f]E_\rho[g]. \end{aligned}$$

PROOF. We prove the case when both f and g increase, as the other case is similar. Observe, as P_ρ is a product measure, that P_ρ has positive correlations in the sense of the FKG inequality (see [7], II.2). Write $\psi(\eta) = E_\eta[g(\eta(t))] = (T_t g)(\eta)$ and note, by the basic coupling, that $\psi(\eta)$ is an increasing function of η. The result follows by applying the FKG inequality to f and ψ, and stationarity of P_ρ. □

Let f and g be local increasing functions, and let $\vec{\boldsymbol{v}}(t)$ and $\vec{\boldsymbol{v}}^*(t)$ be the vectors,

$$\vec{\boldsymbol{v}}(t) = \left(\int_0^t f(\eta(r))\,ds, \int_0^t g(\eta(r))\,ds\right),$$

$$\vec{\boldsymbol{v}}^*(t) = \left(\int_0^t f(\eta^*(r))\,ds, \int_0^t g(\eta^*(r))\,ds\right),$$

where $\eta^*(\cdot) = \eta(t - \cdot)$ is the reversed process from time t.

PROPOSITION 2.1. *With respect to P_ρ, $\vec{\boldsymbol{v}}(t)$ has stationary and weakly positive associated increments.*

PROOF. Stationary increments follows as P_ρ is an equilibrium measure. Compute now, following Theorem 2 of [3] for ϕ and ψ increasing,

$$\begin{aligned}
&E_\rho[\psi(\vec{\boldsymbol{v}}(s))\phi(\vec{\boldsymbol{v}}(t+s) - \vec{\boldsymbol{v}}(s))]\\
&\quad = E_\rho[\psi(\vec{\boldsymbol{v}}(s))E_{\eta(s)}[\phi(\vec{\boldsymbol{v}}(t))]]\\
&\quad = E_\rho[\psi(\vec{\boldsymbol{v}}^*(s))E_{\eta^*(0)}[\phi(\vec{\boldsymbol{v}}(t))]]\\
&\quad = \int E^*_{\eta^*(0)}[\psi(\vec{\boldsymbol{v}}^*(s))]E_{\eta^*(0)}[\phi(\vec{\boldsymbol{v}}(t))]]\,dP_\rho(\eta^*(0))
\end{aligned}$$

by conditioning on $\eta(s)$ and reversing time from s.

Note, as both f and g increase as functions of η, that the factors, $E^*_\eta[\psi(\vec{\boldsymbol{v}}^*(s))]$ and $E_\eta[\phi(\vec{\boldsymbol{v}}(t)]$, as functions of η, also increase with $\eta^*(0)$ by the basic coupling. Therefore, by the previous lemma, the right-hand side above is bounded below by

$$\int E^*_{\eta^*(0)}[\psi(\vec{\boldsymbol{v}}^*(s))]\,dP_\rho(\eta^*(0)) \cdot \int E_{\eta^*(0)}[\phi(\vec{\boldsymbol{v}}(t))]\,dP_\rho(\eta^*(0)),$$

which, when unraveled by reversing time again, becomes $E_\rho[\psi(\mathbf{v}(s))]$ $E_\rho[\phi(\mathbf{v}(t))]$. This completes the proof of the proposition. □

3. Some variance norm estimates. In this section, we develop notions of H_1 and H_{-1} spaces and prove several technical estimates. Some complementary aspects of these spaces are discussed in [6].

First, observe that L is self-adjoint if and only if p is symmetric. In general, however, L may be decomposed as $L = -S - A$ where $-S = (L + L^*)/2$

and $-A = (L - L^*)/2$ denote the symmetric and antisymmetric parts, respectively. Note that $-S$ by itself generates symmetric exclusion with jump rates $(p(j-i)+p(i-j))/2$ and so is P_ρ-reversible. In addition, $-S$ has nonpositive spectrum and, explicitly for local ϕ,

$$(-S\phi)(\eta) = \tfrac{1}{2}\sum_{i,j}(p(j-i) + p(i-j))(\phi(\eta^{i,j}) - \phi(\eta)).$$

Note, in addition, that the quadratic form $< f, (-L)f >_\rho$ is called the Dirichlet form with respect to $-L$. Observe that only the symmetric part S survives in the form and in fact, for local ϕ, one exactly computes

$$\begin{aligned}\langle \phi, (-L)\phi\rangle_\rho &= \langle \phi, S\phi\rangle_\rho \\ &= \tfrac{1}{4}\sum_{i,j} E_\rho[(p(j-i) + p(i-j))(\phi(\eta^{i,j}) - \phi(\eta))^2].\end{aligned} \tag{3.1}$$

To prepare for the definition of certain resolvent norms, note that the operator $(\lambda - L)^{-1}\colon L^2(P_\rho) \to L^2(P_\rho)$, $(\lambda - L)^{-1}f(\eta) = \int_0^\infty e^{-\lambda t}T_t f(\eta)\,dt$, for $\lambda > 0$, is a bounded operator. Also, let $[(\lambda - L)^{-1}]_s$ denote the symmetric part of $(\lambda - L)^{-1}$ and observe that

$$\begin{aligned}[[(\lambda - L)^{-1}]_s]^{-1} &= (\lambda - L^*)(\lambda + S)^{-1}(\lambda - L) \\ &= (\lambda + S) - A(\lambda + S)^{-1}A.\end{aligned}$$

For $f \in L^2(P_\rho)$, the quadratic form $< f, (\lambda - L)^{-1}f >_\rho$ may be written in terms of the semigroup T_t,

$$\langle f, (\lambda - L)^{-1}f\rangle_\rho = \int_0^\infty e^{-\lambda t}\langle f, T_t f\rangle_\rho\,dt, \tag{3.2}$$

or in variational form over local ϕ,

$$\begin{aligned}&\langle f, (\lambda - L)^{-1}f\rangle_\rho \\ &\quad\langle f, [(\lambda - L)^{-1}]_s f\rangle_\rho \\ &\quad= \sup_\phi\{2\langle f, \phi\rangle_\rho - \langle \phi, [[(\lambda - L)^{-1}]_s]^{-1}\phi\rangle_\rho\} \\ &\quad= \sup_\phi\{2\langle f, \phi\rangle_\rho - \langle \phi, (\lambda + S)\phi\rangle_\rho - \langle A\phi, (\lambda + S)^{-1}A\phi\rangle_\rho\}.\end{aligned} \tag{3.3}$$

Define, if the limit exists, the quantity $< f, (-L)^{-1}f >_\rho$ by

$$\langle f, (-L)^{-1}f\rangle_\rho = \lim_{\lambda\to 0}\langle f, (\lambda - L)^{-1}f\rangle_\rho.$$

When $-L = S$ is symmetric, we prove in Lemma 3.3 that the the limit exists in the possibly infinite sense and may be written in terms of the semigroup or in variational form,

$$\langle f, S^{-1}f\rangle_\rho = \int_0^\infty \langle f, T_t f\rangle_\rho \, dt$$

$$= \sup_\phi \{2\langle f, \phi\rangle_\rho - \langle \phi, S\phi\rangle_\rho\}.$$

However, when $-L = S + A$ is asymmetric, it is not clear the limit exists without additional assumptions on the asymmetries.

Standard Dirichlet form spaces may be defined for symmetric p when $-L = S$: let the Hilbert space $H_1(S)$ be the completion with respect to the Dirichlet form $\langle f, Sf\rangle_\rho$,

$$H_1(S) = \text{completion of } \{\phi \text{ local:} < \phi, S\phi >_\rho < \infty\}$$

with norm $\|f\|_1(S) = \sqrt{\langle f, Sf\rangle_\rho}$ and inner-product by polarization.

Let $H_{-1}(S)$ denote the completed dual Hilbert space with respect to $H_1(S)$,

$$H_{-1}(S) = \text{completion of } \{\phi \text{ local:} < \phi, S^{-1}\phi >_\rho < \infty\},$$

with norm $\|f\|_{-1}(S) = \sqrt{\langle f, S^{-1}f\rangle_\rho}$.

For each $\lambda > 0$, dual Hilbert spaces $H_1(\lambda, -L)$ and $H_{-1}(\lambda, -L)$ may also be defined in terms of completions with respect to local functions ϕ of the corresponding norms,

$$\|\phi\|_1(\lambda, L) = \sqrt{\langle(\lambda - L)\phi, (\lambda + S)^{-1}, (\lambda - L)\phi\rangle_\rho}$$

$$= \sqrt{\langle \phi, (\lambda + S)\phi\rangle_\rho + \langle A\phi, (\lambda + S)^{-1}A\phi\rangle_\rho},$$

$$\|\phi\|_{-1}(\lambda, L) = \sqrt{\langle \phi, (\lambda - L)^{-1}\phi\rangle_\rho}.$$

Observe that these norms and spaces make sense for exclusion type operators $-L = S' + A''$ where $-S'$ and $-A''$ are the symmetric and anti-symmetric parts of exclusion generators L' and L'', respectively. We will use analogous notation, $\|\cdot\|_1(\lambda, S' + A'')$ and $\|\cdot\|_{-1}(\lambda, S' + A'')$, to denote $H_1(\lambda, S' + A'')$ and $H_{-1}(\lambda, S' + A'')$ norms.

Recall now the definition of the limiting variance $\sigma^2(f)$ (1.2) if it exists.

LEMMA 3.1. *Let $f(\eta) \in L^2(P_\rho)$ be an increasing nondegenerate function with mean zero, $E_\rho[f] = 0$. Then both $\sigma^2(f)$ and $\langle f, (-L)^{-1}f\rangle_\rho$ exist. Further,*

$$\sigma^2(f) = \lim_{t\to\infty} t^{-1}\sigma_t^2(f)$$

$$= 2\int_0^\infty \langle T_t f, f\rangle_\rho \, dt = 2\langle f, (-L)^{-1}f\rangle_\rho > 0.$$

PROOF. From Lemma 2.2, $\langle f, T_s f\rangle_\rho \geq 0$. Note also, as $\langle f, f\rangle_\rho > 0$, that $\langle f, T_s f\rangle_\rho > 0$ for small s. Compute now, by stationarity of P_ρ, that

$$t^{-1}\sigma_t^2(f) = 2\int_0^t (1 - s/t)E_\rho[f(\eta(0))T_s f(\eta(0))]\,ds.$$

Also note, from (3.2) and the definition of $< f, (-L)^{-1}f >_\rho$, that

$$\langle f, (-L)^{-1}f\rangle_\rho = \lim_{\lambda\to 0}\int_0^\infty e^{-\lambda t}\langle f, T_t f\rangle_\rho\,dt.$$

The lemma now follows from monotone convergence. □

LEMMA 3.2. *Let $f = f_+ - f_-$ where $f_+, f_- \in L^2(P_\rho)$ are increasing functions such that $\sigma^2(f_+), \sigma^2(f_-) < \infty$. Then both $\sigma^2(f) < \infty$ and $\langle f, (-L)^{-1}f\rangle_\rho < \infty$ exist, and also*

$$\sigma^2(f) = 2\int_0^\infty \langle T_t f, f\rangle_\rho\,dt = 2\langle f, (-L)^{-1}f\rangle_\rho.$$

PROOF. Write $t^{-1}\sigma_t^2(f)$ as

$$2\int_0^t (1 - s/t)[\langle f_+, T_t f_+\rangle_\rho\,ds - \langle f_+, T_t f_-\rangle_\rho - \langle f_-, T_t f_+\rangle_\rho + \langle f_-, T_t f_-\rangle_\rho]\,ds.$$

Observe also that

$$\begin{aligned} t^{-1}E_\rho&\left[\int_0^t f_\pm(\eta(s))\,ds\cdot\int_0^t f_\mp(\eta(s))\,ds\right] \\ &= 2\int_0^t (1 - s/t)\langle f_\pm, T_s f_\mp\rangle_\rho\,ds \leq t^{-1}\sigma_t(f_+)\sigma_t(f_-). \end{aligned}$$

The proof now follows from arguments of the previous lemma and $\sigma^2(f_\pm) < \infty$. □

Evidently, proving $\langle f, T_s f\rangle_\rho \geq 0$, for a given f is sufficient to show that $\sigma^2(f) = 2\langle f, (-L)^{-1}f\rangle_\rho$ exists. However, it is not clear that this is true in general. For symmetric systems, though, this positivity holds for all f.

LEMMA 3.3. *Let $f \in L^2(P_\rho)$ be a non-degenerate mean-zero function, $E_\rho[f] = 0$. For symmetric exclusion processes, the limiting variance $\sigma^2(f)$ exists. In fact, $\sigma^2(f) = 2\langle f, S^{-1}f\rangle_\rho > 0$ and may be expressed in terms of the semigroup or in the variational form,*

$$\begin{aligned}\sigma^2(f) &= 2\int_0^\infty \langle f, T_t f\rangle_\rho\,dt \\ &= 2\sup\{2\langle f, \phi\rangle_\rho - \langle \phi, S\phi\rangle_\rho : \phi \text{ local}\}.\end{aligned}$$

PROOF. To show $\sigma^2(f)$ exists and $\sigma^2(f) = 2\int_0^\infty \langle f, T_t f\rangle_\rho dt = 2\langle f, S^{-1}f\rangle_\rho > 0$, observe, from reversibility, that T_t is self-adjoint and

$$\langle f, T_t f\rangle_\rho = \langle T_{t/2} f, T_{t/2} f\rangle_\rho > 0.$$

The arguments of Lemma 3.1 now apply.

To establish the variational formula, we show upper and lower bounds between

$$\langle f, S^{-1}f\rangle_\rho \quad \text{and} \quad V(f) := \sup\{2\langle f, \phi\rangle_\rho - \langle \phi, S\phi\rangle_\rho \colon \phi \text{ local}\}.$$

For the upper bound, note that

$$\begin{aligned}\langle f, (\lambda + S)^{-1} f\rangle_\rho &= \sup\{2\langle f, \phi\rangle_\rho - \lambda\langle \phi, \phi\rangle_\rho - \langle \phi, S\phi\rangle_\rho \colon \phi \text{ local}\}\\ &\le \sup\{2\langle f, \phi\rangle_\rho - \langle \phi, S\phi\rangle_\rho \colon \phi \text{ local}\}.\end{aligned}$$

For the lower bound, let

$$V(f;\phi) := 2\langle f, \phi\rangle_\rho - \langle \phi, S\phi\rangle_\rho.$$

Assume now that $V(f) < \infty$ and fix a local function ϕ_ε, for $\varepsilon > 0$, such that $V(f;\phi_\varepsilon) \ge V(f) - \varepsilon$. Choose $\lambda_\varepsilon > 0$ small enough so that for $0 \le \lambda \le \lambda_\varepsilon$,

$$2\langle f, \phi_\varepsilon\rangle_\rho - \lambda\langle \phi_\varepsilon, \phi_\varepsilon\rangle_\rho - \langle \phi_\varepsilon, S\phi_\varepsilon\rangle_\rho \ge V(f;\phi_\varepsilon) - \varepsilon.$$

Optimizing over local functions ϕ, this implies that $< f, (\lambda+S)^{-1} f >_\rho \ge V(f) - 2\varepsilon$, for arbitrary $\varepsilon > 0$ and we are done.

When $V(f) = \infty$, a similar optimization proves the result. □

We restate a portion of the proved part of Theorem 1.1 in the context of finite variances.

LEMMA 3.4. *Consider the finite-range irreducible symmetric exclusion process. Let f be a local function. Then,*

$$\begin{aligned}\|f\|_{-1}(S) < \infty \Leftrightarrow \sigma_t^2(f) = O(t)&\\ \Leftrightarrow \text{ equivalent conditions in (2.5) hold.}&\end{aligned}$$

PROOF. The first statement follows from the previous lemma. The last statement derives from the equivalence (2.5) and the proved part of Theorem 1.1 for symmetric p (Theorem 1.1 of [14]). □

A useful domination of the $\|\cdot\|_{-1}(\lambda, -L)$ norms by those with respect to the symmetrized systems is given below.

LEMMA 3.5. *For $f \in L^2(P_\rho)$ and $\lambda > 0$,*

$$\|f\|_{-1}(\lambda, -L) \le \|f\|_{-1}(\lambda, S) \le \|f\|_{-1}(S).$$

PROOF. Note in (3.3) that the "$-\langle A\phi, (\lambda+S)^{-1}A\rangle_\rho$" term can be dropped to give the first bound. Dropping the "$-\lambda\langle\phi, S\phi\rangle_\rho$" term gives the second upper bound, noting the variational formula in Lemma 3.3. □

We will say that operators S and S' have equivalent Dirichlet (quadratic) forms if there are constants $0 < C_1 < C_2$ such that for all local functions ϕ,

$$C_1\langle\phi, S'\phi\rangle_\rho \le \langle\phi, S\phi\rangle_\rho \le C_2\langle\phi, S'\phi\rangle_\rho.$$

The following results will be useful in making some comparisons.

LEMMA 3.6. *Let $L = -S - A$ and $L' = -S' - A$ be generators with the same anti-symmetric part. If S and S', with respect to some constants $0 < C_1 < C_2$, have equivalent Dirichlet forms, then for $f \in L^2(P_\rho)$ and $\lambda > 0$ we have*

$$\begin{aligned}[\max(C_1^{-1}, C_2)]^{-1}\|f\|_{-1}^2(C_1^{-1}\lambda, -L') &\le \|f\|_{-1}^2(\lambda, -L)\\ &\le [\min(C_1, C_2^{-1})]^{-1}\|f\|_{-1}^2(C_2^{-1}\lambda, -L').\end{aligned}$$

PROOF. The bounds follow from manipulations of the variational formula (3.3). In the sense of quadratic forms, note the inequalities,

$$C_1(C_1^{-1}\lambda + S') \le (\lambda + S) \le C_2(C_2^{-1}\lambda + S')$$

and

$$C_2^{-1}(C_2^{-1}\lambda + S')^{-1} \le (\lambda + S)^{-1} \le C_1^{-1}(C_1^{-1}\lambda + S')^{-1}.$$

Plugging into (3.3) and observing $C_2^{-1} \le C_1^{-1}$, we have that $\|f\|_{-1}^2(\lambda, -L)$ is bounded above by

$$\begin{aligned}&\sup_\phi\{2\langle f, \phi\rangle_\rho - C_1\langle\phi, (C_1^{-1}\lambda + S')\phi\rangle_\rho - C_2^{-1}\langle A\phi, (C_2^{-1}\lambda + S')^{-1}A\phi\rangle_\rho\}\\
&\quad\le \sup_\phi\{2\langle f, \phi\rangle_\rho - C_1\langle\phi, (C_2^{-1}\lambda + S')\phi\rangle_\rho - C_2^{-1}\langle A\phi, (C_2^{-1}\lambda + S')^{-1}A\phi\rangle_\rho\}\\
&\quad\le \sup_\phi\{2\langle f, \phi\rangle_\rho - \min(C_1, C_2^{-1})\langle\phi, (C_2^{-1}\lambda + S')\phi\rangle_\rho\\
&\qquad\qquad - \min(C_1, C_2^{-1})\langle\phi, (C_2^{-1}\lambda + S')\phi\rangle_\rho\}.\end{aligned}$$

Taking the supremum now over $\psi = \sqrt{\min(C_1, C_2^{-1})}\phi$ gives the upper bound in the lemma. The lower bound follows similarly. □

Let $e^l \in \mathbb{Z}^d$, for $l = 1, \ldots, d$, denote the unit vectors on the positive axes.

LEMMA 3.7. *Let $-S$ and $-S'$ generate symmetric exclusion processes on $\mathbb{Z}^d$ with finite-range irreducible jump rates p and p', respectively. Then $-S$ and $-S'$ have equivalent Dirichlet forms.*

PROOF. Let $-S_1$ denote the generator of nearest-neighbor symmetric exclusion on $\mathbb{Z}^d$ with jump rates p_1 satisfying $p_1(e^l) = p_1(-e^l) = (2d)^{-1}$ for $l = 1, \dots, d$ and $p_1(i) = 0$ otherwise. It is enough to show that the quadratic forms of $-S$ and $-S'$ are each equivalent to those for $-S_1$.

For the upper bound, recall the Dirichlet form computation (3.1). For fixed $i = (i_1, \dots, i_d)$ and $j = (j_1, \dots, j_d)$, let $k^l = (j_1, \dots, j_l, i_{l+1}, \dots, i_d)$ for $l = 1, \dots, d$ and $k^0 = i$. Define the nearest-neighbor rectangular path r^l from k^{l-1} to k^l, for $l = 1, \dots, d$, as follows: let $r^l(0) = k^{l-1}$, and if $i_l \le j_l$, let $r^l(m) = r^l(m-1) + e^l$; otherwise, let $r^l(m) = r^l(m-1) - e^l$, for $m = 1, \dots, |j_l - i_l|$.

Observe that the configuration $\eta^{i,j}$, which transposes site values at i and j, can also be obtained in a two step procedure: First, transpose forward along the nearest-neighbor pair sequence $(r^l(0), r^l(1)), \dots, (r^l(|j_l - i_l| - 1), r^l(|j_l - i_l|))$ successively for $l = 1, 2, \dots, d$ (this has the effect of moving the value at i to j). Second, transpose back along the reversed sequence beginning with the penultimate pair, $(r^d(|j_d - i_d| - 1), r^d(|j_d - i_d| - 2)), \dots, (r^d(1), r^d(0))$, and then along $(r^{d-l}(|j_{d-l} - i_{d-l}|), r^{d-l}(|j_{d-l} - i_{d-l}| - 1)), \dots, (r^{d-l}(1), r^{d-l}(0))$ for $l = 1, \dots, d-1$ (this brings the value at j to i while shifting back the other values to their original positions). Let $D(i, j) = \sum_{l=1}^d |j_l - i_l|$. Then by transposition invariance of P_ρ and Schwartz inequality, we have that

$$E_\rho\big[(\phi(\eta^{i,j}) - \phi(\eta))^2\big] \le 2D(i, j) \sum_{l=1}^{d} \sum_{m=0}^{|j_l - i_l| - 1} E_\rho\big[(\phi(\eta^{r^l(m), r^l(m+1)}) - \phi(\eta))^2\big].$$

The upper bound $\langle \phi, S\phi \rangle_\rho \le C_2 \langle \phi, S_1 \phi \rangle_\rho$ now follows from the fact that p is finite range.

For the lower bound, note that $\langle \phi, S_1\phi \rangle_\rho = (1/d) \sum_i \sum_{l=1}^d E_\rho[(\phi(\eta^{i,i+e_l}) - \phi(\eta))^2]$. Also note, as p is irreducible, that there exists a finite path r^l from the origin 0 to e_l, $0 = r^l(0), r^l(1), \dots, r^l(n_l)$, for $l = 1, \dots, d$, in the support of p such that a p-random walker may jump from 0 to $r^l(1)$, then to $r^l(1) + r^l(2)$, and so on to $r^l(1) + \cdots + r^l(n_l) = e_l$. Denote $j^l(k) = r^l(0) + \cdots + r^l(k)$ to simplify notation, and bound

$$E_\rho[(\phi(\eta^{i,i+e_l}) - \phi(\eta))^2] \le n_l \sum_{m=0}^{n_l - 1} E_\rho[(\phi(\eta^{i+j^l(m), i+j^l(m+1)}) - \phi(\eta))^2].$$

The bound $\langle \phi, S_1\phi \rangle_\rho \le C_1^{-1} \langle \phi, S\phi \rangle_\rho$ follows once again as p is finite range. $\square$

Resolvent equations also yield some bounds on the variance $\sigma^2(f)$. Let $f \in L^2(P_\rho)$ be a mean-zero function, and write, for $\lambda > 0$,

$$\lambda u^L_{\lambda,f}(\eta) - L u^L_{\lambda,f}(\eta) = f(\eta) \quad \text{where } u^L_{\lambda,f}(\eta) = (\lambda - L)^{-1} f(\eta). \tag{3.4}$$

Multiplying through by $u_\lambda = u^L_{\lambda,f}$ and taking P_ρ-expectation, we obtain

$$\lambda\langle u_\lambda, u_\lambda\rangle_\rho + \|u_\lambda\|_1^2(S) = \langle f, u_\lambda\rangle_\rho. \tag{3.5}$$

LEMMA 3.8. *Suppose $\|f\|_{-1}(S) < \infty$. Then both $\lambda\langle u_\lambda\ u_\lambda\rangle_\rho \le \|f\|_{-1}^2(S)$ and $\|u_\lambda\|_1^2(S) \le \|f\|_{-1}^2(S)$.*

PROOF. Note that by the Schwarz inequality, $\langle f, u_\lambda\rangle_\rho \le \|f\|_{-1}(S)$ $\|u_\lambda\|_1(S)$. Therefore from (3.5), $\|u_\lambda\|_1^2(S) \le \|f\|_{-1}(S)\|u_\lambda\|_{-1}(S)$ and so $\|u_\lambda\|_{-1}(S) \le \|f\|_{-1}(S)$. The inequality on $\lambda < u_\lambda, u_\lambda >_\rho$ now also follows from (3.5). □

LEMMA 3.9. *Let $t > 0$ and $f \in L^2(P_\rho)$ be a mean-zero function. Then*

$$\sigma_t^2(f) = t\langle f, u_{t^{-1}}\rangle_\rho + O(\langle u_{t^{-1}}, u_{t^{-1}}\rangle_\rho) \quad and$$

$$t^{-1}\sigma_t^2(f) \le 10\langle f, u_{t^{-1}}\rangle_\rho.$$

Also, for f such that $\|f\|_{-1}(S) < \infty$, we have $t^{-1}\sigma_t^2(f) \le 10\|f\|_{-1}(S)$.

PROOF. Calculate

$$\begin{aligned}\int_0^t f(\eta(s))\,ds &= M_\lambda(t) + (u_\lambda(\eta(0)) - u_\lambda(\eta(t))) + \lambda\int_0^t u_\lambda(\eta(s))\,ds\\ &= M_\lambda(t) + I_2 + I_3,\end{aligned}$$

where $M_\lambda(t)$ is the martingale, $M_\lambda(t) = u_\lambda(\eta(t)) - u_\lambda(\eta(0)) - \int_0^t Lu_\lambda(\eta(s))\,ds$, with quadratic variation, by stationarity, $2E_\rho\big[\int_0^t u_\lambda(-L)u_\lambda\,ds\big] = 2t\|u_\lambda\|_1^2(S)$. Bound the second and third terms as follows:

$$E_\rho[I_2^2] \le 2\langle u_\lambda, u_\lambda\rangle_\rho \quad \text{and} \quad E_\rho[I_3^2] \le \lambda^2 t^2\langle u_\lambda, u_\lambda\rangle_\rho.$$

To prove the first and second claims of the lemma, choose $\lambda = t^{-1}$ and plug into (3.5). The last claim now follows from the previous lemma. □

4. Variance bounds: mean-zero *p*. We collect together some variance estimates which, in particular, will be used to prove the variance bounds with respect to mean-zero p in Theorem 1.1.

We begin with two facts proved in [15].

LEMMA 4.1. *Let $f \in L^2(P_\rho)$ be a nondegenerate function such that $\|f\|_{-1}(S) < \infty$. Then, with respect to simple exclusion with finite-range, mean-zero, irreducible p, we have that the limits $\sigma^2(f)$ and $\langle f, (-L)^{-1}f\rangle_\rho$ exist and $\sigma^2(f) = 2\langle f, (-L)^{-1}f\rangle_\rho < \infty$.*

This lemma is proved in Section 4 of [15].

LEMMA 4.2. *Let ϕ and ψ be local functions. For simple exclusion processes with finite-range, mean-zero, irreducible p, there is a constant $C = C(d, p, \rho) \geq 1$ so that a sector condition holds*:

$$\langle \phi, (-L)\psi\rangle_\rho \leq C\langle \phi, S\phi\rangle_\rho^{1/2}\langle \psi, S\psi\rangle_\rho^{1/2}. \tag{4.1}$$

The proof is Theorem 5.1 of [15].

Note that when L is symmetric, C may be taken to be $C = 1$. A corollary of the last lemma is the following.

LEMMA 4.3. *Let $L = -S - A$ correspond to finite-range, mean-zero, irreducible p. With respect to local functions ϕ, $\lambda \geq 0$, and C as in* (4.1),

$$\langle A\phi, (\lambda + S)^{-1}A\phi\rangle_\rho \leq (C+1)^2\langle \phi, (\lambda + S)\phi\rangle_\rho.$$

PROOF. Write $-L = S + A$ and observe from Lemma 4.2 and the Schwarz inequality, for test functions ϕ and ψ, that

$$\begin{aligned} |\langle A\phi, \psi\rangle_\rho| &\leq |\langle \phi, S\psi\rangle_\rho| + C\sqrt{\langle \phi, S\phi\rangle_\rho\langle \psi, S\psi\rangle_\rho} \\ &\leq (C+1)\sqrt{\langle \phi, S\phi\rangle_\rho\langle \psi, S\psi\rangle_\rho}. \end{aligned}$$

Note that this inequality carries over, by approximation, to functions ϕ and ψ in $H_1(S)$. Fix now a local ϕ and note that $E_y[A\phi] \equiv 0$ for all y so that, by (2.5) and Lemma 3.4, $A\phi \in H_{-1}(S)$. Consequently, $\psi = (\lambda + S)^{-1}A\phi \in H_1(\lambda, S) \subset H_1(S)$. Applying the inequality to ϕ and ψ, we obtain

$$\begin{aligned} \langle A\phi, (\lambda + S)^{-1}A\phi\rangle_\rho &\leq (C+1)\sqrt{\langle \phi, S\phi\rangle_\rho}\sqrt{\langle A\phi, (\lambda + S)^{-2}SA\phi\rangle_\rho} \\ &\leq (C+1)\sqrt{\langle \phi, (\lambda + S)\phi\rangle_\rho}\sqrt{\langle A\phi, (\lambda + S)^{-1}A\phi\rangle_\rho}. \end{aligned}$$

To finish, we divide through by the last factor. □

The next lemma shows that the $H_{-1}(\lambda, -L)$ and $H_{-1}(\lambda, S)$ norms, for mean-zero p, are equivalent. This will imply nondegeneracy of $\sigma^2(f)$ for $f \in H_{-1}(S)$.

LEMMA 4.4. *Let $L = -S - A$ correspond to finite-range, mean-zero, irreducible p and let $f \in L^2(P_\rho)$. There exists $0 < C \leq 1$ such that for $\lambda > 0$*,

$$C\langle f, (\lambda + S)^{-1}f\rangle_\rho \leq \langle f, (\lambda - L)^{-1}f\rangle_\rho \leq \langle f, (\lambda + S)^{-1}f\rangle_\rho.$$

In particular, the forms $\langle f, (-L)^{-1}\rangle_\rho$ and $\langle f, S^{-1}f\rangle_\rho$ are equivalent. As a consequence, when $\|f\|_{-1}(S) < \infty < \infty$, $\sigma^2(f) > 0$.

PROOF. The upper bound was established in Lemma 3.5. The lower bound follows by an application of Lemma 4.3 to the bound below the third term in (3.3). The form equivalence follows by taking $\lambda \downarrow 0$ and Lemma 4.1. Hence, the last statement follows from Lemmas 4.1 and 3.3. □

The following Tauberian result will be used to obtain lower bounds. Some of the intuition in the argument is owed to Lemma 2.4 in [1].

LEMMA 4.5. *Let $U(t)$ be a nonnegative function and let $V(t) = \int_0^t U(s)\,ds$. Suppose for all small $\lambda > 0$, some $\alpha \in [0,1]$ and $0 < C_1, C_2 < \infty$, that $V(t) \le C_1 t^{\alpha+1}$ (respectively, greater than or equal to $-C_1 t \log t$) and also that $\int_0^\infty e^{-\lambda t} U(t) dt \ge C_2 \lambda^{-\alpha-1}$ (respectively, greater than or equal to $-C_2\lambda^{-1}\log\lambda$). Then, for some $0 < C_3 < \infty$ and all large t, we have that*

$$V(t) \ge C_3 t^{\alpha+1} \text{ (respectively, greater than or equal to } C_3 t \log t).$$

PROOF. We prove the assertion for the case $V(t) \le C_1 t^{\alpha+1}$ and $\int_0^\infty e^{-\lambda t} U(t)\,dt \ge C_2 \lambda^{-\alpha-1}$ as the other case is similar. If the claim is not true, then $\liminf V(t) t^{-(\alpha+1)} = 0$, so that for each fixed $\varepsilon > 0$, a subsequence $\{t_n\}$, $t_n \uparrow \infty$, exists where $V(t_n) t_n^{-(\alpha+1)} \le \varepsilon$. Now let K be so large that $(C_1/\alpha + 1)\int_K^\infty e^{-s} s^{\alpha+1}\,ds \le C_2/2$, and note from the assumptions on U, that $V(t)$ increases and $\int_0^\infty e^{-\lambda t} V(t)\,dt \ge C_2 \lambda^{-\alpha-2}$. As a consequence of these remarks, we have

$$\begin{aligned} C_2 \lambda^{-\alpha-2} &\le \int_0^{t_n} e^{-\lambda t} V(t)\,dt + \int_{t_n}^\infty e^{-\lambda t} V(t)\,dt \\ &\le \varepsilon t_n^{\alpha+2} + C_1 \int_{t_n}^\infty e^{-\lambda t} t^{\alpha+1}\,dt \\ &= \varepsilon t_n^{\alpha+2} + C_1 \lambda^{-\alpha-2} \int_{\lambda t_n}^\infty e^{-s} s^{\alpha+1}\,ds. \end{aligned}$$

Multiplying by $\lambda^{\alpha+2}$, we obtain that $C_2 \le \varepsilon(\lambda t_n)^{\alpha+2} + (C_1/\alpha+1)\int_{\lambda t_n}^\infty e^{-s} s^{\alpha+1}\,ds$. With $\lambda = K t_n^{-1}$, the last estimate reduces to $C_2/2 \le \varepsilon (K+1)^{\alpha+2}$ which yields a contradiction for ε small enough. □

Let $u_f(t) = \langle f, T_t f\rangle_\rho$, $U_f(t) = \int_0^t u_f(s)\,ds$, and $V_f(t) = \int_0^t U_f(s)\,ds$. To apply the last lemma, we identify the variance $\sigma_t^2(f)$ in terms of the kernel $u_f(t)$.

LEMMA 4.6. *For $f \in L^2(P_\rho)$, $\sigma_t^2(f) = 2V_f(t)$.*

PROOF. Observe that $\sigma_t^2(f) = 2\int_0^t (t-s)\langle f, T_s f\rangle_\rho\,ds$. The lemma now follows from integration by parts. □

LEMMA 4.7. *Let L generate an exclusion process with finite-range, mean-zero, irreducible p. Then there exists $C = C(p(\cdot)) > 0$ such that for $0 < \lambda < 1$,*

$$C\lambda^{-1/2} \le \langle f, (\lambda - L)^{-1} f\rangle_\rho \le C\lambda^{-1/2} \qquad \text{for } f = C_0^\rho \text{ in } d = 1 \text{ and}$$

$$-C\log\lambda \le \langle f, (\lambda - L)^{-1} f\rangle_\rho \le -C^{-1}\log\lambda \quad \text{for } f = C_0^\rho \text{ in } d = 2 \text{ and}$$

$$f = C_{(0,1)}^\rho \text{ in } d = 1.$$

PROOF. Recall that we proved in Lemma 4.4, for $f \in L^2$ and some constant $0 < C \le 1$, that

$$C\langle f, (\lambda + S)^{-1} f\rangle_\rho \le \langle f, (\lambda - L)^{-1} f\rangle_\rho \le \langle f, (\lambda + S)^{-1} f\rangle_\rho.$$

Let now $-S_1$ and T_t^1 denote the generator and semigroup, for the $d = 1$ nearest-neighbor symmetric system with $p(1) = p(-1) = 1/2$, and observe, from irreducibility assumptions on p and Lemma 3.7, that S and S_1 have equivalent Dirichlet forms. Consequently, from Lemma 3.6, we have that there exists $0 < C, D, E < \infty$ such that

$$C\langle f, (D\lambda + S_1)^{-1} f\rangle_\rho \le \langle f, (\lambda + S)^{-1} f\rangle_\rho \le C^{-1}\langle f, (E\lambda + S_1)^{-1} f\rangle_\rho.$$

We now show appropriate bounds on $\langle f, (\lambda - S_1)^{-1} f\rangle_\rho$ to complete the proof. With $u_f^1(t) = \langle f, T_t^1, f\rangle_\rho$ and $U_f^1(t) = \int_0^t u_f^1(s)\,ds$, rewrite $\langle f, (\lambda + S_1)^{-1} f\rangle_\rho$, from (3.2) and integration by parts, as

$$\begin{aligned}\langle f, (\lambda + S_1)^{-1} f\rangle_\rho &= \int_0^\infty e^{-\lambda t} u_f^1(t)\,dt \\ &= \lambda \int_0^\infty e^{-\lambda t} U_f^1(t)\,dt.\end{aligned}$$

Now note, for the one-point function $f = C_0^\rho$ that $u_f^1(t) = \rho(1-\rho)P_{0,0}(t) \sim t^{-d/2}$ from duality (2.6) and local limit estimates. For the two-point function $f = C_{(0,1)}^\rho$ in $d = 1$, note first that $u_f^1(t) = \rho^2(1-\rho)^2 P_{(01),(01)}^{SE}(t)$ from duality (2.6) and second that it is proved in Section 3 of [14] that $U_f^1(t) \approx \log(t)$. Compute therefore that $\langle C_0^\rho, (\lambda + S_1)^{-1} C_0^\rho\rangle_\rho \approx \lambda^{-1/2}$ in $d = 1$ and $-\log(\lambda)$ in $d = 2$ as $\lambda \to 0$. Also deduce that $\langle C_{(0,1)}^\rho, (\lambda + S_1)^{-1} C_{(0,1)}^\rho\rangle_\rho \approx -\log(\lambda)$ in $d = 1$ as $\lambda \to 0$. This completes the proof. $\square$

We calculate now divergence rates of $\sigma_t^2(C_0^\rho)$ in $d = 1, 2$ and $\sigma_t^2(C_{(0,1)}^\rho)$ in $d = 1$.

LEMMA 4.8. *Let L generate simple exclusion with finite-range, mean-zero, irreducible rates p. Then there exists a constant $C = C(\rho, d, p(\cdot)) > 0$ such that for large t,*

$$Ct^{3/2} \le \sigma_t^2(C_0^\rho) \le C^{-1} t^{3/2} \text{ in } d = 1 \quad \text{and}$$

$$Ct\,\log(t) \le \sigma_t^2(C_0^\rho) \le C^{-1} t\,\log(t) \text{ in } d = 2.$$

Also,

$$Ct\,\log(t) \le t^{-1}\int_0^t \sigma_s^2(C^\rho_{(0,1)}) \quad \textit{and} \quad \sigma_t^2(C^\rho_{(0,1)}) \le C^{-1}t\,\log(t) \textit{ in } d=1.$$

In fact, when p is symmetric, in $d=1$,

$$Ct\,\log(t) \le \sigma_t^2(f) \le C^{-1}t\,\log(t).$$

PROOF. The upper bounds follow from Lemma 3.9 and Lemma 4.7.

The lower bounds are proved in two parts, first for C_0^ρ and second for $C^\rho_{(0,1)}$. It will be helpful to recall, for $f \in L^2$, that $\langle f, (\lambda - L)^{-1}f\rangle_\rho$ equals

$$\begin{aligned}\int_0^\infty e^{-\lambda t}u_f(t)\,dt &= \lambda\int_0^\infty e^{-\lambda t}U_f(t)\,dt\\ &= \lambda^2\int_0^\infty e^{-\lambda t}V_f(t)\,dt.\end{aligned}$$

Also recall from Lemma 4.6 that $V_f(t) = \sigma_t^2(f)/2$.

For the case $f = C_0^\rho$, observe that $u_f(t) \ge 0$ from Lemma 2.2, and therefore $U_f(t) \ge 0$. Hence, the lower bounds for $\sigma_t^2(C_0^\rho)$ in $d = 1, 2$ follow from Lemma 4.5 with $U(t) = U_f(t)$, Lemma 4.7 and the above observations combined with the proven upper bounds.

For the case $f = C^\rho_{(0,1)}$ in $d = 1$, let $W_f(t) = \int_0^t V_f(s)\,ds$ and note from the observations above and the proved upper bound that $V_f(t) \ge 0$ and $W_f(t) \le 2C^{-1}t^2\log(t)$ for large t. Then the lower bound $Ct\log(t) \le t^{-1}W_f(t) = (2t)^{-1}\int_0^t \sigma_s^2(f)\,ds$ follows from Lemma 4.5 with $U(t) = V_f(t)$ and Lemma 4.7.

Finally, to prove the lower bound $Ct\,\log(t) \le \sigma_t^2(C^\rho_{(0,1)})$ in $d = 1$ when p is symmetric, we use a different technique. Note that the lower bound would follow from the first statement of Lemmas 3.9 and 4.7 provided we show that

$$\langle u^{-S}_{t^{-1},f}, u^{-S}_{t^{-1},f}\rangle_\rho = O(t). \tag{4.2}$$

To this end, observe in the symmetric case, that $L = -S$ and $T_t = T_t^*$. Write

$$\begin{aligned}\langle u^{-S}_{\lambda,f}, u^{-S}_{\lambda,f}\rangle_\rho &= \int_0^\infty\int_0^\infty \exp(-\lambda(r+s))\langle T_rf, T_sf\rangle_\rho\,ds\,dr\\ &= \int_0^\infty\int_0^\infty \exp(-\lambda(r+s))\langle T_{r+s}f, f\rangle_\rho\,ds\,dr\\ &= \int_0^\infty \exp(\lambda^2 s - \lambda - 1)(-\lambda s)V_f(s)\,ds.\end{aligned}$$

The estimate (4.2) now follows from Lemma 4.6 and the proved upper bound with $\lambda = t^{-1}$. $\square$

5. Proofs of the main theorems. The main idea of the proofs of Theorems 1.1 and 1.2 is to divide the work into a "variance part and a "weak convergence" part. The variance work, relies on the bases representations (2.3) and (2.4) and on negative norm bounds of Sections 3 and 4. The weak convergence, typically however, follows from Theorem 2.1 for associated random vectors or extant proofs, under the assumption of the variance estimates.

PROOF OF THEOREM 1.1(i). Let $f(\eta)$ be a local function such that the mean does not vanish, $E_\rho[f] \neq 0$. One computes that $\sigma_t^2(f) \geq (E_\rho[f]t)^2$ and $\langle f, (\lambda - L)^{-1} f\rangle_\rho \geq \lambda^{-1}(E_\rho[f])^2$, by substituting $\phi(\eta) = \lambda^{-1}E_\rho[f]$ into (3.3), so that the theorem in this case follows. From now on, let $f(\eta)$ be a local mean-zero function, $E_\rho[f] = 0$.

We show now that $\sigma_t^2(f) = O(t)$ in $d \geq 3$. Observe that if $\sigma_t^2(C_I^\rho) = O(t)$ for all $I \subset \mathbb{Z}^d$ such that $|I| = k$ and $k \geq 1$, then the estimate on $\sigma_t^2(f)$ follows from the finite centered basis expansion for f (2.3) and the Schwarz inequality. But, the desired bounds follow from Lemmas 3.9 and 3.4.

We now prove that $\sigma^2(f) = 2\langle f, (-L)^{-1} f\rangle_\rho < \infty$ exists and is finite. By applying the monotone basis decomposition (2.4) to f, denote $f_+(\eta)$ and $-f_-(\eta)$ as the sums of components of f corresponding to positive and negative coefficients, respectively. Note that both f_+ and f_- are local increasing mean-zero functions where $f = f_+ - f_-$ and $\limsup t^{-1}\sigma_t^2(f_+) < \infty$ and $\limsup t^{-1}\sigma_t^2(f_-) < \infty$. In addition, also note from Lemma 3.1 that both $\sigma^2(f_+) = 2\langle f_+, (-L)^{-1} f_+\rangle_\rho < \infty$ and $\sigma^2(f_-) = 2\langle f_-, (-L)^{-1} f_-\rangle_\rho < \infty$ exist. The identification and finiteness of $\sigma^2(f)$ now follows from Lemma 3.2.

At this point, observe that the statement for $d \leq 2$ also follows from the last paragraph.

Finally, when f increases, note that $\sigma^2(f) > 0$ from Lemma 3.1. Also, in $d \geq 3$, note that, for local mean-zero f, $\sigma^2(f) > 0$ follows from the methods of Lemma 4.4 (to prove positivity for mean-zero systems) which use, instead of (4.1), sector type conditions proved in Sections 5 and 6 of [13].

We now show the invariance principle in the theorem. Define $\vec{v}(t) = (v_1(t), v_2(t))$ where

$$v_1(t) = \int_0^t f_+(\eta(s))\,ds \quad \text{and} \quad v_2(t) = \int_0^t f_-(\eta(s))\,ds.$$

Note, as f_- and f_+ are bounded functions, that $\vec{v}(t) \in C[0, \infty)$. Also, observe that $\int_0^t f(\eta(s))\,ds = v_1(t) - v_2(t)$ and, from Proposition 2.1, that $\vec{v}(t)$ has stationary and weakly positive associated increments with respect to P_ρ. Therefore, by Theorem 2.1, given that the covariance condition (2.1) for $\vec{v}(t)$ holds, the invariance principle follows.

To prove the covariance condition for $\vec{v}(t)$, note that we have already established

$$\lim_{t\to\infty} t^{-1} E_\rho[v_1(t)^2] = \sigma^2(f_+) < \infty \quad \text{and} \quad \lim_{t\to\infty} t^{-1} E_\rho[v_2(t)^2] = \sigma^2(f_-) < \infty.$$

Note in addition, from monotone convergence, that

$$\lim_{t\to\infty} t^{-1}E_\rho[v_1(t)v_2(t)] = \lim_{t\to\infty}\int_0^t 2(1-s/t)E_\rho[f_+(\eta(0))f_-(\eta(s))]\,ds$$

exists as $E_\rho[f_+(\eta(0))f_-(\eta(s))] \geq 0$ from Lemma 2.2. Finiteness of $\lim t^{-1} E_\rho[v_1(t)v_2(t)]$ follows from the Schwarz inequality.

This completes the proof of (i) in Theorem 1.1. □

PROOF OF OF THEOREM 1.1(ii) AND PROPOSITION 1.1. Let f be a local function. The case when $E_\rho[f] \neq 0$ is handled as in the proof of part (i). So, in the following, we assume f is mean-zero, $E_\rho[f] = 0$.

We first prove the variance estimates. Note, for those functions satisfying conditions (1.3), that $0 < \sigma^2(f) = 2\langle f, (-L)^{-1}\rangle_\rho < \infty$ exists and is finite and positive from Lemmas 4.1, 4.4 and 3.4. Observe also, from Lemma 2.1, that all local mean-zero functions f which violate (1.3) may be put in the form

$$f(\eta) = \alpha C_0^\rho(\eta) + \beta C_{(0,1)}^\rho(\eta) + \bar{f} \quad \text{for } d = 1$$

and

$$f = \alpha C_0^\rho(\eta) + \bar{f} \quad \text{for } d = 2,$$

where both α and β cannot vanish in $d = 1$, $\alpha \neq 0$ in $d = 2$ and $\bar{f}$ satisfies conditions (1.3) so that $\sigma^2(\bar{f}) < \infty$ from Lemmas 4.1 and 3.4.

Note, from Lemma 4.8, that $\sigma_t^2(C_0^\rho) \approx t^{3/2}$ and $t\log(t)$ in $d = 1$ and 2, respectively, and $0 < \liminf(t^{-1}\int_0^t \sigma_s^2(C_{(0,1)}^\rho)\,ds)/(t\log(t))$ and $\sigma_t^2(C_{(0,1)}^\rho) = O(t\log(t))$ in $d = 1$. Also note, in the case p symmetric, the stronger bound $0 < \liminf(\sigma_t^2(C_{(0,1)}^\rho))/(t\log(t))$. The bounds in Proposition 1.1 now follow straightforwardly.

Now observe, for f which violates (1.3), that $\langle f, (-L)^{-1}f\rangle_\rho = \infty$ from Lemmas 4.4 and 3.4. Therefore, we deduce from the above remarks, for these functions, that

$$\limsup_{t\to\infty} t^{-1}\sigma_t^2(f) = 2\langle f, (-L)^{-1}f\rangle_\rho = \infty.$$

We now prove the invariance principle in (2), Theorem 1.1. Note that the invariance principle is proved in Section 4 of [15] for those functions $f \in H_{-1}(S)$, with variance $\sigma^2(f) = 2\langle f, (-L)^{-1}f\rangle_\rho$. Therefore, as $\sigma^2(f) < \infty$ when f satisfies (1.3) $\Leftrightarrow f \in H_{-1}(S)$ (Lemma 3.4), we obtain the invariance principle in the theorem.

This completes the proof of (2), Theorem 1.1 and Proposition 1.1. □

PROOF OF THEOREM 1.2. Note first that the central limit theorem, for fixed t, $\lim_{\alpha\to\infty}\beta(d,\alpha)^{-1/2}\int_0^{\alpha t} C_0^\rho(\eta(s))\,ds = N(0, \sigma^2(\rho, d)t)$, is Kipnis's theorem [4].

Convergence of finite-dimensional distributions follows also from the approach used in [4]. In fact, for the case $d = 2$, straightforward successive conditioning of the martingale representation in Section 3.2 of [4] gives this convergence. However, for the case $d = 1$, the martingale representation in Section 3.3 of [4] is used to prove the finite-dimensional convergence to $B_{3/4}(\sigma^2(\rho,1)t)$.

What remains is to show tightness of the finite-dimensional distributions in the uniform topology. Note that, in equilibrium P_ρ, additive functionals with respect to $f(\eta) = \eta_0 - \rho$, as f increases, possess stationary and weakly positive associated increments from Proposition 2.1 and therefore satisfy the demimartingale assumption of Corollary 6 of [10]. Tightness of these additive functionals now follows by applying the maximal inequality in Corollary 6 of [10], noting that $\sigma_t^2(C_0^\rho) \sim \beta(d,t)$. $\square$

6. Application to second-class particles. Let η' be a configuration drawn from $P_\rho\{\cdot|\eta_0 = 0\}$ and let η be such that

$$(\eta)_j = \begin{cases} 1, & \text{if } k = 0, \\ \eta'_k, & \text{if } k \neq 0. \end{cases}$$

That is, η is the configuration which places a particle at the origin.

The basic coupling ensures that we may couple two copies of the exclusion process starting from configurations η and η' so that at a later time t, the states $\eta(t)$ and $\eta'(t)$ also differ exactly at one coordinate. Let $R(t)$ be the position of this discrepancy or second-class particle at time t. The position $R(\cdot)$ is not generally Markovian with respect to its own history and in fact has been connected with movement of "shocks" in the η' system [11]. The larger joint process $(R(t), \eta(t))$, however, is Markovian with joint generator,

$$\begin{aligned}\bar{L}f(r,\eta) = &\sum_{i,j\neq r} \eta_i(1-\eta_j)(f(r,\eta^{i,j}) - f(r,\eta))p(i,j) \\ &+ \sum_j (p(r-j)\eta_j + p(j-r)(1-\eta_j))(f(j,\eta^{r,j}) - f(r,\eta)).\end{aligned}$$

From this expression, we see when p is symmetric, $R(t)$ is a random walk. For all other p however, R depends on the environment. Observe now, under equilibrium P_ρ, that R should displace by k on average with "homogenized" jump rate $(1-\rho)p(k) + \rho p(-k)$. Note, in particular, if $\rho = 1/2$, then R should behave as a symmetric random walk. See [2] in this regard.

Consider now the following calculations:

$$\lim_{t\to\infty} E_\rho \left\{ \frac{1}{\sqrt{t}} \int_0^t (\eta_0(s) - \rho)\, ds \right\}^2 = 2\int_0^\infty E_\rho[\eta_0(s)\eta_0(0)] - (E_\rho[\eta_0])^2\, ds.$$

The integrand is further analyzed,

$$\begin{aligned}&E_\rho[\eta_0(s)\eta_0(0)] - (E_\rho[\eta_0])^2 \\ &\quad = \rho(1-\rho)\{E_\rho[\eta_0(s) = 1|\eta_0(0) = 1] - E_\rho[\eta_0(s) = 1|\eta_0(0) = 0]\} \\ &\quad = \rho(1-\rho)\bar{P}_\rho\{R(s) = 0\},\end{aligned}$$

where $\bar{P}_\rho$ is the coupled measure. Hence, the variance of the occupation time at the origin is $2\rho(1-\rho)$ times the expected occupation time at the origin of the second-class particle.

We say that $R(t)$ is P_ρ-*transient* or P_ρ-*recurrent* if, respectively,

$$\int_0^\infty \bar{P}_\rho\{R(s)=0\}\,ds < \infty \text{ or } = \infty.$$

A consequence of the variance bounds in Proposition 1.1 applied to $f(\eta) = \eta_0 - \rho$ is the following.

COROLLARY 6.1. *Let p be a finite-range jump rate whose symmetrization is irreducible. Then $R(t)$ is P_ρ-recurrent in dimensions $d = 1, 2$ when $\sum ip(i) = 0$. However, in $d \geq 3$, $R(t)$ is P_ρ-transient.*

It is natural now to speculate, given the homegenization discussion above, that in $d = 1, 2$ when $\sum ip(i) \neq 0$ that $R(t)$ is P_ρ-transient if and only if $\rho \neq 1/2$. This motivates the conjecture after Theorem 1.1.

Finally, some information on the recurrence strength of $R(t)$, when p is mean-zero in $d = 1, 2$, is as follows.

COROLLARY 6.2. *Let p be a finite-range mean-zero irreducible jump rate. There exist constants $0 < C = C(d, \rho, p) < \infty$ such for all t large,*

$$C\sqrt{t} \leq \int_0^t \bar{P}_\rho\{R(s)=0\}ds \leq C^{-1}\sqrt{t} \quad \textit{in } d = 1$$

and

$$C\log t \leq \int_0^t \bar{P}_\rho\{R(s)=0\}ds \leq C^{-1}\log t \quad \textit{in } d = 2.$$

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