Diffusive Limit of a Tagged Particle in Asymmetric Simple Exclusion Processes

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Dedicated to Dr C. R. Rao on his eightieth birthday.

Abstract

Invariance principles are proved under diffusive scaling for the centered position of a tagged particle in the simple exclusion process with asymmetric nonzero drift jump probabilities in dimensions $d \geq 3$. The method of proof is by martingale techniques which rely on the fact that symmetric random walks are transient in high dimensions. © 2000 John Wiley & Sons, Inc.

1 Introduction

In the study of interacting particle systems, an important component is the motion of a tagged particle as it moves through other particles that are viewed as indistinguishable among themselves. An initial step is to understand the behavior of a tagged particle in equilibrium, in particular, the fluctuation behavior. The simple exclusion processes are a natural class of models where one can study this problem. In the case of simple exclusion processes on $\mathbb{Z}^d$ when the underlying random walk has a symmetric distribution, this question is answered in [1] for $d = 1$ nearest-neighbor walks, and in [6] for all other distributions and dimensions. It was generalized to the case of random walks with mean 0 in [11]. When the mean is nonzero, the only case studied previously is the asymmetric nearest-neighbor $d = 1$ case in [5]. Here we consider the general case where the mean is not zero. We need to assume, however, that *the dimension d of the lattice $\mathbb{Z}^d$ is at least* 3.

If X is a countable set and $p(x,y)$ is the transition probability matrix of a Markov chain, then one can define the simple exclusion process on X with transition probabilities $p(\cdot,\cdot)$ on X. This is actually a Markov process whose state space is the set $\Omega(X)$ of all possible subsets of X. The set $A \subset X$ signifies that there are particles present at every point of A and that the sites in $X \setminus A$ are empty. One can also

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view $\Omega(X)$ as the set of maps $\eta : X \to \{0,1\}$ where $\eta(x) = 1$ means that $x \in A$ or that there is a particle at the site x. $\eta(x) = 0$ signifies that the site is empty. Each particle waits for an exponential time and then jumps to a new site or at least tries to. The new site y is selected with probability $p(x,y)$ where the current site is x. If the site y is empty, then the jump is completed and things start afresh. If the site is occupied with a particle already there, the jump is forbidden and the original particle remains at x. Again things start afresh. All the particles are doing this simultaneously, and because we are dealing with continuous time, there will be no ties to resolve. However, it requires some work to make sure that the process is well-defined, especially if there are an infinite number of particles to begin with.

Suppose $u : \Omega \to \mathbb{R}$ is a function that depends only on a finite number of coordinates, i.e., on $\{\eta(x) : x \in F\}$ for some finite set F; then the infinitesimal generator of the process is defined by

$$(\mathcal{L}u)(\eta) = \sum_{x,y} \eta(x)(1-\eta(y))p(x,y)[u(\eta^{x,y}) - u(\eta)]$$

where

$$\eta^{x,y}(z) = \begin{cases} \eta(z) & \text{if } z \neq x \text{ or } y \\ \eta(y) & \text{if } z = x \\ \eta(x) & \text{if } z = y\,. \end{cases} \tag{1.1}$$

Under some mild conditions on $p(\cdot,\cdot)$ (for instance, it is sufficient to assume that $\sup_y \sum_x p(x,y) < \infty$), there is a well-defined stochastic process starting from any initial configuration in Ω. The details can be found in [8]. This is referred to as the simple exclusion process on X with transition probability $p(\cdot,\cdot)$. A special case of particular interest is when $p(\cdot,\cdot)$ is doubly stochastic, i.e.,

$$\sum_x p(x,y) = 1 \ \text{ for all } y \quad \text{in addition to} \quad \sum_y p(x,y) = 1 \ \text{ for all } x\,.$$

In this case the process is always well-defined. A more special case is when $p(\cdot,\cdot)$ is symmetric. This case is referred to as symmetric simple exclusion.

In the doubly stochastic case, the uniform measure on X is a σ-finite invariant measure for the Markov chain. One can verify that, for any $0 \le \rho \le 1$, the Bernoulli product measure P_ρ on Ω defined by $P_\rho[\eta(x) = 1] = \rho$ for every x, with $\{\eta(x)\}$ being mutually independent for different x, is an ergodic invariant measure for the evolution on Ω with generator L. If $p(\cdot,\cdot)$ is symmetric, the evolution is reversible with respect to each P_ρ and has for its Dirichlet form the quantity

$$D^\rho(u) = \frac{1}{2} \int_\Omega \sum_{x,y} p(x,y)[u(\eta^{x,y}) - u(\eta)]^2 dP_\rho\,.$$

Our interest is mainly in the case where $X = \mathbb{Z}^d$ for some $d \ge 3$ and $p(x,y) = p(y-x)$ for some probability distribution $p(\cdot)$ on $\mathbb{Z}^d$. We can assume without loss of generality that $p(0) = 0$. We are interested in the case when

$$m = \sum_x x p(x) \neq 0 . \tag{1.2}$$

We shall assume for simplicity that $p(x) = 0$ outside a finite set F, although it will not matter that much. We shall start the process in equilibrium, that is to say, with the initial distribution being some P_ρ but conditioned to have a particle at the origin 0, which will be tagged. As the system evolves, we wish to follow the trajectory of the tagged particle. It is convenient to change our coordinates in $\mathbb{Z}^d$ so that the tagged particle is always seen at the origin. In other words, our description of the current state consists of the position of the tagged particle, which we denote by z, and the environment seen from the tagged particle, which can be viewed as a point in $\Omega_0 = \{\eta : \mathbb{Z}^d \setminus \{0\} \to \{0,1\}\}$.

There are two types of motion. When an untagged particle jumps, it is from some x to a y, neither of which can be 0. When the tagged particle jumps from 0 to an empty site x, the origin shifts with it, so what we see is a shift of the environment by $-x$. The tagged particle at 0 is not part of the shift, and so we always end up with $-x$ being empty. In other words, if we define on the set $\eta(x) = 0$, the map τ_x,

$$(\tau_x \eta)(y) = \begin{cases} \eta(x+y) & \text{for } y \neq 0,\ y \neq -x \\ 0 & \text{for } y = -x, \end{cases}$$

then the generator of our process is given by $\mathcal{L} = \mathcal{L}^{\text{sh}} + \mathcal{L}^{\text{ex}}$ where

$$(\mathcal{L}^{\text{sh}} u)(z,\eta) = \sum_x (1-\eta(x)) p(x) [u(z+x, \tau_x \eta) - u(z,\eta)]$$

and

$$(\mathcal{L}^{\text{ex}} u)(z,\eta) = \sum_{x,y \neq 0} \eta(x)(1-\eta(y)) p(y-x) \left[u(z, \eta^{x,y}) - u(z,\eta)\right] .$$

The environment by itself is a Markov process, and the generator is given by $L = L^{\text{sh}} + L^{\text{ex}}$ where

$$(L^{\text{sh}} u)(\eta) = \sum_x (1-\eta(x)) p(x) [u(\tau_x \eta) - u(\eta)] \tag{1.3}$$

$$(L^{\text{ex}} u)(\eta) = \sum_{x,y \neq 0} \eta(x)(1-\eta(y)) p(y-x) \left[u(\eta^{x,y}) - u(\eta)\right] . \tag{1.4}$$

We have adopted the convention that the generators associated with the original process are denoted by script $\mathcal{L}$, while those associated with the environment process are denoted by L. The Bernoulli product measure P_ρ restricted to points in $\mathbb{Z}^d \setminus \{0\}$ is an ergodic invariant measure for the environment process (see proposition 3 of [9]). An elementary computation shows that

$$\mathcal{L} z = g(\eta) = \sum_x x p(x)(1-\eta(x)) \tag{1.5}$$

so that

$$z(t) - z(0) - \int_0^t g(\eta(s))ds = M(t)$$

is a martingale with stationary increments. One can see almost surely [9],

$$\lim_{t\to\infty} \frac{z(t)}{t} = \lim_{t\to\infty} \frac{1}{t} \int_0^t g(\eta(s))ds = \int_{\Omega_0} g(\eta)dP_\rho = m(1-\rho)\,.$$

Let

$$\xi(t) = \frac{1}{\sqrt{t}}(z(t) - m(1-\rho)t)\,; \tag{1.6}$$

in this article we prove a functional central limit theorem for ξ. As remarked earlier, the symmetric case for nearest-neighbor $d = 1$ walks was considered in [1], and all the other symmetric cases in [6]. The more general case $m = 0$ was covered in [11]. Also, the $d = 1$ asymmetric nearest-neighbor $m \neq 0$ case was done in [5]. We are now interested in the case when $m \neq 0$ in high dimensions, which requires different methods. Our main result is the following theorem:

THEOREM 1.1 *As $\alpha \to \infty$, the distribution of $(1/\sqrt{\alpha})(z(\alpha t) - m(1-\rho)\alpha t)$ converges weakly in Skorohod space to a nondegenerate Brownian motion with covariance $\mathbf{C}(\rho)$ given in formula* (2.23)*.*

In the next section, we outline the proof of this theorem. Our proof follows the approach in [6] of considering the environment process as seen by the tagged particle and to explore the associated martingales. The key step in this approach is an estimate of the resolvent equation. The methods used here for solving the resolvent equation are similar to the methods used for solving the fluctuation-dissipation equation of the hydrodynamical limit of the simple exclusion processes in [7] and of lattice gases in [2]. Notice that both the fluctuation-dissipation equation and the resolvent equation require estimates on the Green's function of the generator, though in somewhat different contexts. More technical comments can be found at the end of the next section.

2 Outline of the Proof of Theorem 1.1

The tagged particle process has a lot of martingales associated with it. They are of the form

$$M_x(t) = N_x(t) - \int_0^t p(x)(1-\eta(s,x))ds \tag{2.1}$$

and

$$M_{x,y}(t) = N_{x,y}(t) - \int_0^t p(y-x)\eta(s,x)(1-\eta(s,y))ds\,, \tag{2.2}$$

corresponding to the number of jumps $N_x(t)$ of size x for the tagged particle or the number of jumps $N_{x,y}(t)$ of untagged particles from x to y. The compensators of their squares are given by

$$d\langle [M_{x,y}(t)]^2\rangle = p(y-x)\eta(t,x)(1-\eta(t,y))dt\,,$$
$$d\langle [M_x(t)]^2\rangle = p(x)(1-\eta(t,x))dt\,.$$

These are the basic martingales, and every other martingale is a combination of these. For example, the position $z(t)$ of the tagged particle satisfies

$$z(t) = \sum_x xN_x(t) = \sum_x xM_x(t) + \int_0^t \sum_x xp(x)(1-\eta(s,x))ds\,.$$

Subtracting the term $m(1-\rho)t$, we have

$$\xi(t) = z(t) - m(1-\rho)t = \xi^{(1)}(t) + \int_0^t g(\eta(s))ds \tag{2.3}$$

where

$$\xi^{(1)}(t) = \sum_x xM_x(t) \tag{2.4}$$

and g is the vector-valued function given by (1.5), i.e.,

$$g(\eta) = \sum_x xp(x)(\rho - \eta(x))\,.$$

The problem now reduces to proving the central limit theorem for the centered additive functional

$$A(t) = \int_0^t g(\eta(s))ds\,.$$

We would like to do it by the martingale method, which means finding a square integrable martingale $\xi^{(2)}(t)$ such that

$$A(t) = \xi^{(2)}(t) + B(t)$$

with $B(\cdot)$ becoming negligible under rescaling. Then the central limit theorem for square integrable martingales with stationary increments applies to the sum $\xi^{(1)}(t) + \xi^{(2)}(t)$, and this will establish the result. We may write the generator of the process that describes the environment of the tagged particle as the sum of its symmetric and antisymmetric pieces,

$$L = L_{\text{sym}} + L_{\text{skew}}\,.$$

Associated with the symmetric part, we have the Dirichlet form

$$D^\rho(u) = D^{ex,\rho}(u) + D^{sh,\rho}(u) \tag{2.5}$$

where

$$D^{ex,\rho}(u) = \frac{1}{2}\int_{\Omega_0} \sum_{x,y\neq 0} a(y-x)[u(\eta^{x,y}) - u(\eta)]^2 dP_\rho \tag{2.6}$$

and

$$D^{sh,\rho}(u) = \frac{1}{2}\int_{\Omega_0} \sum_x a(x)(1-\eta(x))[u(\tau_x\eta)-u(\eta)]^2 dP_\rho \tag{2.7}$$

with $a(x) = \frac{1}{2}[p(x)+p(-x)]$. The associated Dirichlet norm is $\|\cdot\|_1 = \sqrt{D_\rho(u)}$. The dual norm $\|\cdot\|_{-1}$ is defined by

$$\|g\|_{-1} = \inf\left[C : \left|\int gu\,dP_\rho\right| \le C\|u\|_1 \text{ for all local functions } u\right].$$

We define $\mathbb{H}_1$ and $\mathbb{H}_{-1}$ as the Hilbert spaces generated by local functions with respect to the norms $\|\cdot\|_1$ and $\|\cdot\|_{-1}$. We shall drop the index ρ, which is fixed throughout the whole paper.

2.1 Step 1: $\mathbb{H}_{-1}$ Estimate and Tightness

Our first step is to show that for the function g defined in (1.1), $\|g\|_{-1} < \infty$. This follows from the next lemma.

LEMMA 2.1 *Let f be a mean zero local function, i.e., a function that depends only on a finite number of coordinates. Then for $d \ge 3$, there is a bound*

$$|\langle f, v\rangle| \le C_f \|v\|_1 .$$

Lemma 2.1 will be proved in Section 3. Recall that $\sqrt{t}\,\xi$ is a sum of a martingale $\xi^{(1)}$ and $A(t)$. The martingale $\xi^{(1)}$ satisfies Doob's inequality. Hence the tightness of ξ in Skorohod space is deduced from the following general theorem:

THEOREM 2.2 *Let P be a stationary and ergodic Markov process with marginal μ on some state space. Let A and A^* be the infinitesimal generators in $L_2(\mu)$ of the original process and its time reversal, respectively. Let $S = \frac{1}{2}[A + A^*]$ be the symmetrized operator defined on* $\operatorname{dom}(A) \cap \operatorname{dom}(A^*)$ *with the associated Dirichlet form*

$$D(u) = \langle -Su, u\rangle . \tag{2.8}$$

Let f be a function on the state space belonging to $L_2(\mu)$ such that for some sequence $u_n \in$ $\operatorname{dom}(A)\cap\operatorname{dom}(A^*)$ *we have*

$$\|Su_n - f\|_{L_2(\mu)} \to 0$$

while

$$\sup_n D(u_n) \le C . \tag{2.9}$$

Then for all $T \ge 0$

$$E^P\left\{\sup_{0\le t\le T}\left|\int_0^t f(x(s))ds\right|^2\right\} \le 8CT$$

with the same constant C as in equation (2.9).

PROOF: We deal with both the forward and backward filtrations.

$$u_n(x(t)) - u_n(x(s)) - \int_s^t (Au_n)(x(\sigma))d\sigma = M_n^+(t) - M_n^+(s)$$

are martingales adapted to the forward filtration for $t \geq s$.

$$u_n(x(s)) - u_n(x(t)) + \int_s^t (A^* u_n)(x(\sigma))d\sigma = M_n^-(s) - M_n^-(t)$$

are martingales adapted to the backward filtration for $s \leq t$. In any case

$$\int_s^t (Su_n)(x(\sigma))d\sigma = \frac{1}{2}[M_n^+(t) - M_n^+(s) + M_n^-(s) - M_n^-(t)]\,.$$

Since

$$E^P\{[M_n^+(t) - M_n^+(s)]^2\} = E^P\{[M_n^-(t) - M_n^-(s)]^2\} \leq 2C|t-s|\,,$$

Doob's inequality applied separately to the two martingales provides an estimate that is uniform in n. Since $Su_n \to f$ in $L_2(\mu)$, we are done. $\square$

Remark. If $\mathrm{dom}(A) \cap \mathrm{dom}(A^*)$ is large enough to be a core for the Dirichlet space, from an estimate of the form

$$|\langle f, u\rangle| \leq \sqrt{C}\sqrt{D}(u)$$

valid for all $u \in \mathrm{dom}(A) \cap \mathrm{dom}(A^*)$ we can construct by standard resolvent techniques a sequence u_n with the desired properties, with the same exact constant C appearing in equation (2.9). In our case of tagged versions of simple exclusion processes, the space of local functions, included in all domains, is always large enough to form a core.

2.2 Step 2: Resolvent Estimate

Consider the resolvent equation

$$\lambda u_\lambda - Lu_\lambda = f\,. \tag{2.10}$$

The key input for the proof of Theorem 1.1 is the following estimate for the solution of the resolvent equation (2.10) to be proved in Section 6.

THEOREM 2.3 *For any local f with mean 0 and $d \geq 3$,*

$$\sup_\lambda \|Lu_\lambda\|_{-1} < \infty\,, \qquad \sup_\lambda \|\lambda u_\lambda\|_{-1} < \infty\,. \tag{2.11}$$

2.3 Step 3: Approximation via the Resolvent Equation

THEOREM 2.4 *If f is a local function with mean 0, then for the solution u_λ of the resolvent equation we have*

$$\lim_{\lambda\to 0} \|u_\lambda - w\|_1 = 0 \tag{2.12}$$

for some $w \in \mathbb{H}_1$ and

$$\lim_{\lambda\to 0} \lambda\|u_\lambda\|_0^2 = 0\,. \tag{2.13}$$

PROOF: Multiplying the resolvent equation by u_λ and integrating, we get

$$\lambda\|u_\lambda\|_0^2+\|u_\lambda\|_1^2=\langle u_\lambda,f\rangle\,. \tag{2.14}$$

From the estimate of Lemma 2.1, we see that $\sup_{0\le\lambda\le 1}\|u_\lambda\|_1<\infty$. Hence we can choose a subsequence $\lambda_n\to 0$ such that along this subsequence $u_{\lambda_n}=u_n$ has a weak limit w in the Dirichlet space $\mathbb{H}_1$. Because $\lambda\|u_\lambda\|_0^2$ is bounded, $\lambda u_\lambda\to 0$ as $\lambda\to 0$ in $L_2=\mathbb{H}$ and consequently, from equation (2.11), converges to 0 in $\mathbb{H}_{-1}$. Together with the resolvent equation, this implies that Lu_n converges weakly to $-f$ in $\mathbb{H}_{-1}$. By standard functional analysis there are convex combinations v_n of $u_1,\dots,u_n$ such that v_n and Lv_n converge strongly to w and $-f$ in $\mathbb{H}_1$ and $\mathbb{H}_{-1}$, respectively. It is easy to see, from equation (2.11), that $\|v_n\|_1\to\|w\|_1$ and $\langle v_n,Lv_n\rangle$ converges to $-\langle w,f\rangle$, thus proving

$$\langle w,f\rangle=\|w\|_1^2\,.$$

From equation (2.14), we have, in particular,

$$\limsup_{n\to\infty}\|u_{\lambda_n}\|_1^2\le\limsup_{n\to\infty}[\lambda_n\|u_{\lambda_n}\|_0^2+\|u_{\lambda_n}\|_1^2]=\langle w,f\rangle=\|w\|_1^2\,. \tag{2.15}$$

Since $\|\cdot\|_1$ is lower semicontinuous and $u_{\lambda_n}\to w$ weakly,

$$\|w\|_1^2\le\liminf_{n\to\infty}\|u_{\lambda_n}\|_1^2\,.$$

We have thus proved that the equality holds in equation (2.15). This in turns implies that u_{λ_n} converges to w strongly. The proof of Theorem 2.4 is completed by showing that the limits w_1 and w_2 along any two subsequences λ_n and λ_n' are necessarily equal. To see this, we observe that $L(u_{\lambda_n}-u_{\lambda_n'})\to 0$ weakly in $\mathbb{H}_{-1}$ while $u_{\lambda_n}-u_{\lambda_n'}\to w_1-w_2$ strongly in $\mathbb{H}_1$, proving that

$$\|u_{\lambda_n}-u_{\lambda_n'}\|_1^2=-\langle L(u_{\lambda_n}-u_{\lambda_n'}),(u_{\lambda_n}-u_{\lambda_n'})\rangle\to 0\,.$$

□

We now have the following martingale decomposition theorem:

THEOREM 2.5 *There is a square integrable martingale $M(t)$ with stationary increments and an additive functional $\Omega(t)$ such that*

$$\int_0^t f(\eta(s))ds=M(t)+\Omega(t)$$

with

$$T^{-1}EM(T)^2\le\|f\|_{-1}^2 \tag{2.16}$$

and

$$\lim_{T\to\infty}\frac{1}{T}E\left[[\Omega(T)]^2\right]=0\,. \tag{2.17}$$

PROOF: The proof is based on Theorem 2.4. From Ito's formula

$$u_\lambda(\eta(t)) - u_\lambda(\eta(0)) = \int_0^t Lu_\lambda(\eta(s))ds + M_\lambda(t),$$

where M_λ denotes the martingale part with quadratic variation given by $EM_\lambda(t)^2 = t\|u_\lambda\|_1^2$. From the resolvent equation (2.10),

$$\int_0^t f(\eta(s))ds = M_\lambda(t) + \Omega_\lambda(t)$$

where

$$\Omega_\lambda(t) = u_\lambda(\eta(t)) - u_\lambda(\eta(0)) - \int_0^t \lambda u_\lambda(\eta(s))ds.$$

By Theorem 2.4, the martingale part converges as $\lambda \to 0$, to, say, $M(t)$. Clearly, $M(t)$ satisfies the estimate (2.16). Hence we obtain

$$\int_0^t f(\eta(s))ds = M(t) + \Omega(t)$$

where

$$\Omega(t) = \Omega_\lambda(t) + [M_\lambda(t) - M(t)].$$

If we pick $\lambda = 1/T$, then together with (2.13)

$$\begin{aligned}\frac{1}{T}E^P\left[(\Omega(T))^2\right] &\le \frac{12}{T}\|u_{1/T}\|_0^2 + \frac{4}{T}E^P\left[[M_{1/T}(T) - M(T)]^2\right]\\ &= \frac{12}{T}\|u_{1/T}\|_0^2 + 4E^P\left[[M_{1/T}(1) - M(1)]^2\right] \to 0\end{aligned}$$

as $T \to \infty$. This concludes the proof. □

We can now apply Theorem 2.5 to our setting with f taken to be the function g defined in equation (1.5). Hence

$$\xi(t) = \xi^{(1)}(t) + \xi^{(2)}(t) + \Omega(t) \tag{2.18}$$

where $\xi^{(1)}$ is defined in equation (2.4), $\xi^{(2)}(t)$ is the martingale obtained from applying Theorem 2.5, and $\Omega(t)$ is the error term satisfying equation (2.17). Hence, up to a negligible error, we have two square integrable martingales $\xi_1(t)$ and $\xi_2(t)$ with stationary increments adapted to the environment process. From the martingale convergence theorem (see theorem 3.2 of [3] where condition (b) can be checked in this situation), $\xi(\alpha t)/\sqrt{\alpha}$, as $\alpha \uparrow \infty$, converges to a Brownian motion with the covariance matrix $\mathbf{C}(\rho)$ characterized by

$$\langle \mathbf{C}(\rho)a, a\rangle = E^{P_\rho}\{\langle \xi_1(1) + \xi_2(1), a\rangle^2\}.$$

Hence our final task is to estimate the variance.

2.4 Step 4: Bounds on the Variance

For upper bounds on the variance, we can estimate each term separately, where, as for lower bounds, we have to worry about possible cancellations. We start with the upper bound. From the definition of $\xi^{(1)}$ in (2.4), we have immediately

$$E^{P_\rho}\{\langle \xi_1(1), a\rangle^2\} = (1-\rho)\sum_x p(x)\langle x, a\rangle^2 . \tag{2.19}$$

An upper bound of $\xi^{(2)}$ can also be easily obtained from equation (2.16)

$$E\{\langle \xi_2(1), a\rangle^2\} \le C\rho(1-\rho)\sum_x p(x)\langle x, a\rangle^2 . \tag{2.20}$$

Therefore, we have the upper bound that follows:

THEOREM 2.6 $\langle \mathbf{C}(\rho)a, a\rangle \le C(1-\rho)\sum_x p(x)\langle x, a\rangle^2 .$

The lower bound can be reduced to the symmetric case. The limiting variance of the tagged particle in the asymmetric case is bounded below by the limiting variance of the tagged particle in the corresponding symmetric case. A lower bound for the symmetric case was derived in [10].

THEOREM 2.7 $\langle \mathbf{C}a, a\rangle \ge C_3(1-\rho)\langle a, a\rangle$.

PROOF: The martingale $\xi^{(2)}(t)$ can be expressed in terms of the solution w obtained in Therorem 2.4

$$\xi^{(2)}(t) = \sum_x w_x M_x(t) + \sum_{x,y} w_{x,y} M_{x,y}(t) \tag{2.21}$$

where $\{M_x(\cdot)\}$ and $\{M_{x,y}(\cdot)\}$ are the basic martingales introduced in equations (2.1) and (2.2). The functions $\{w_x(\cdot)\}$ and $\{w_{x,y}(\cdot)\}$ are the limits of $u_{\lambda,x} = (u_\lambda(\tau_x\eta) - u_\lambda(\eta))$ and $u_{\lambda,x,y} = (u_\lambda(\eta^{x,y}) - u_\lambda(\eta))$, respectively, in $L_2(P_\rho)$. The convergence of u_λ to w in $\mathbb{H}_1$ guarantees the convergence to 0 of

$$E^{P_\rho}\left[\sum_x p(x)(1-\eta(x))(u_{\lambda,x} - w_x)^2 + \sum_{x,y\neq 0} p(y-x)(u_{\lambda,x,y} - w_{x,y})^2\right]$$

as $\lambda \to 0$. From (2.4), $\xi^{(1)}$ has the representation

$$\xi^{(1)}(t) = \sum_x x M_x(t) . \tag{2.22}$$

Combining equations (2.21) and (2.22), we get the representation

$$\xi^{(1)}(t) + \xi^{(2)}(t) = \sum_x (w_x + x) M_x(t) + \sum_x w_{x,y} M_{x,y}(t) ,$$

from which we can easily deduce that

$$\langle C(\rho)a, a\rangle = E^{P_\rho}\left[\sum_x p(x)(1-\eta(x))(w_x^a + \langle x, a\rangle)^2 + \sum_{x,y} p(y-x) w_{x,y}^2\right] . \tag{2.23}$$

Here we have denoted by w^a the solution that corresponds to $f = \langle a, g\rangle$. The right-hand side of formula (2.23), being essentially a Dirichlet form, is clearly insensitive to replacing $p(\cdot)$ by its symmetrized version. The corresponding formula for the symmetric case, proved in [6], involves solving a different equation and therefore a different w, but for the symmetric case the correct w is given by a variational formula that minimizes the expression on the right-hand side of formula (2.23). Therefore the quantity $\langle C(\rho)a, a\rangle$ is bounded below by the corresponding value for the symmetric case for which a lower bound of the form claimed in Theorem 2.7 was proved in [10]. $\square$

We have based our proof on estimating the resolvent equation (2.10) with the key estimate equation (2.11). In fact, we can also base it on the following estimate: *For any $\varepsilon > 0$ there is a local function u_ε such that*

$$\|Lu_\varepsilon - f\|_{-1} \le \varepsilon .$$

From this estimate we can obtain a result analogous to Theorem 2.4 ([7] or section 6 of [4]) and thus the martingale decomposition Theorem 2.5. This was the estimate established in [2, 7] for the fluctuation-dissipation equation. In a sense these two estimates on the Green's functions can be substituted for each other in many contexts; see section 6 of the lecture [4].

3 Estimates Related to Simple Exclusion Processes

Simple exclusion processes describe dynamics of infinitely many simple random walks on $\mathbb{Z}^d$ with the exclusion that no two particles are allowed to occupy the same site. The simple random walk on $\mathbb{Z}^d$ has the generator given by

$$(Af)(x) = \sum_y p(y-x)[f(y) - f(x)] .$$

The corresponding symmetric generator is

$$(\mathbf{S}f)(x) = \sum_y a(y-x)[f(y) - f(x)] \tag{3.1}$$

where

$$p(x) = a(x) + b(x) ,$$

$a(\cdot)$ and $b(\cdot)$ being the symmetric and asymmetric components of $p(\cdot)$, respectively. Note that

$$|b(x)| = \frac{1}{2}|p(x) - p(-x)| \le \frac{1}{2}[p(x) + p(-x)] = a(x) .$$

The Dirichlet form is

$$\mathcal{D}(f) = \frac{1}{2}\sum_{x,y} a(y-x)[f(y) - f(x)]^2 . \tag{3.2}$$

Since we are mainly interested in the tagged particle process, the state space is $\mathbb{Z}^d \setminus \{0\}$ rather than $\mathbb{Z}^d$. The random walk on $\mathbb{Z}^d \setminus \{0\}$ can be viewed as the random walk on $\mathbb{Z}^d$ with jumps into the origin disallowed. The main result in this section was obtained in collaboration with C. Landim and was reported in [7] by a purely analytic proof. Here we give a more probabilistic proof and for more general processes.

We can consider the general setting of an irreducible transient Markov process on a countable state space X that is symmetric with respect to the counting or uniform measure on X. The generator and the Dirichlet form are given by (3.1) and (3.2) with obvious interpretation. There is a subset $E \subset X$ that is excluded, and transitions into E are disallowed. On the state space $Y = X \setminus E$ we have the induced generator

$$(\tilde{\mathbf{S}}u)(x) = \sum_{y \in Y} a(x,y)[u(y) - u(x)]$$

for $x \in Y$. The corresponding Dirichlet form is

$$\tilde{\mathcal{D}}(u) = \frac{1}{2} \sum_{x,y \in Y} a(x,y)[u(y) - u(x)]^2.$$

For the special case of a random walk, E consists of just the origin. We shall keep the set E general because we want to apply it to other settings as well. We shall assume that the basic process on X is transient, which means that the Green's function exists, i.e.,

$$g(x,y) = \int_0^\infty p(t,x,y)dt < \infty \quad \text{for all } x,y \in X$$

where $p(t,x,y)$ is the transition probability function. The Green's function $g(x,y)$ satisfies

$$g(x,y) = g(y,x) \le g(x,x) \quad \text{for all } x,y \in X$$

and

$$(\mathbf{S}g(x,\cdot))(y) = \sum_z [g(x,z) - g(x,y)]a(y,z) = -\delta_x(y) \quad \text{for all } x,y \in X$$

where $\delta_x(y) = 1$ if $x = y$ and 0 otherwise.

We want to show that the induced process on Y with generator $\tilde{\mathbf{S}}$ is again transient and compare its Green's function $\tilde{g}(x,y)$ to the original Green's function $g(x,y)$. We will assume that the probability

$$\theta(x) = P_x\{x(t) \text{ visits } E \text{ for some } t \ge 0\} < 1$$

for each $x \in Y$. This is clearly satisfied for any random walk in $d \ge 3$ with $E = \{0\}$. It is well-known that in the transient case there is an estimate of the form

$$|u(x)| \le C(x)\sqrt{\mathcal{D}(u)}$$

valid uniformly for all functions u that vanish outside a finite set and by completion for all functions that belong to the Dirichlet space $\mathbb{H}_1$, which we recall was defined as the completion of the space of finitely supported functions under the norm $\|u\|_1 = \sqrt{\mathcal{D}(u)}$. A precise estimate on the constant $C(x)$ is given in the following lemma:

LEMMA 3.1 *Suppose $V(x)$ is a nonnegative, compactly supported function. Then*

$$\sum_x u^2(x)V(x) \le \sup_x \left[\sum_y g(x,y)V(y)\right]\mathcal{D}(u)$$

for all u. In particular, taking V to be 1 at x and 0 elsewhere,

$$|u(x)| \le \sqrt{g(x,x)}\sqrt{\mathcal{D}(u)}\,.$$

PROOF: Let us define $W(x) = \sum_y g(x,y)V(y)$ and $C = \sup_x W(x)$. Since $0 \le W \le C$ and $V \ge 0$,

$$\begin{aligned}
\sum_x u^2(x)V(x) &\le C\sum_x u^2(x)\frac{V(x)}{W(x)} \\
&= -C\sum_x \frac{u^2(x)}{W(x)}(SW)(x) \\
&= \frac{C}{2}\sum_{x,y}\left[\frac{u^2(y)}{W(y)} - \frac{u^2(x)}{W(x)}\right][W(y)-W(x)]a(x,y) \\
&= \frac{C}{2}\sum_{x,y}\left[u^2(y)+u^2(x)-u^2(x)\frac{W(y)}{W(x)} - u^2(y)\frac{W(x)}{W(y)}\right]a(x,y) \\
&\le \frac{C}{2}\sum_{x,y}[u^2(y)+u^2(x)-2u(x)u(y)]a(x,y) = C\mathcal{D}(u)\,.
\end{aligned}$$

□

Notice that $\theta(x)$ solves the equation

$$\begin{cases} (\mathbf{S}\theta)(x) = \sum_{y\in X} a(x,y)[\theta(y)-\theta(x)] = 0 & \text{for } x\in Y \\ \theta(x) = 1 & \text{for } x \in E\,. \end{cases}$$

Therefore

$$(\tilde{\mathbf{S}}\theta)(x) = -\sum_{y\in E} a(x,y)[\theta(y)-\theta(x)] = -\sum_{y\in E} a(x,y)[1-\theta(x)] = -A(x)(1-\theta(x))$$

where

$$A(x) = \sum_{y\in E} a(x,y)\,. \tag{3.3}$$

Because of irreducibility, $A(x)$ cannot vanish identically on Y. Thus $\theta(x)$ is a bounded, nonconstant superharmonic function, and so the process is transient. We will actually assume that

$$\sup_{x\in Y}\theta(x)=\beta<1\,. \tag{3.4}$$

The following lemma is a quantified version of the transience:

LEMMA 3.2 *For the Green's function $\tilde g(x,y)$, we have*

$$\sum_{y\in Y}\tilde g(x,y)A(y)\le\frac{\theta(x)}{(1-\beta)}\,.$$

PROOF: Since

$$-(\tilde{\mathbf{S}}\theta)(x)\ge(1-\beta)A(x)\,,$$

the lemma follows from the maximum principle. □

LEMMA 3.3 *Let $U\ge 0$ be supported on Y, that is, it vanishes on E. Then*

$$\begin{aligned}\sup_{x\in Y}\sum_{y\in Y}\tilde g(x,y)U(y)&\le\frac{1}{(1-\beta)}\sup_{x\in X}\sum_{y\in X}g(x,y)U(y)\\&=\frac{1}{(1-\beta)}\sup_{x\in Y}\sum_{y\in Y}g(x,y)U(y)\end{aligned} \tag{3.5}$$

Furthermore, for all $x,y\in Y$,

$$\tilde g(x,y)\le g(x,y)+\frac{1}{(1-\beta)}g(y,y)\theta(x) \tag{3.6}$$

and

$$\tilde g(x,x)\le\frac{1}{(1-\beta)}g(x,x)\,. \tag{3.7}$$

PROOF: Let

$$W(x)=\sum_{y\in X}g(x,y)U(y).$$

The function W is nonnegative and solves $\mathbf{S}W=-U$. A computation shows that for $x\in Y$,

$$\begin{aligned}(\tilde{\mathbf{S}}W)(x)&=(\mathbf{S}W)(x)-\sum_{y\in E}a(x,y)[W(y)-W(x)]\\&\le -U(x)+\sum_{y\in E}a(x,y)W(x)=-U(x)+A(x)W(x)\le -U(x)+CA(x)\end{aligned}$$

where $C=\sup_{x\in Y}W(x)$. Then

$$\sum_{y\in Y}\tilde g(x,y)U(y)\le W(x)+C\sum_{y\in Y}\tilde g(x,y)A(y)\le W(x)+\frac{C\theta(x)}{(1-\beta)}\,. \tag{3.8}$$

Taking the supremum over x, we get

$$\sup_{x\in Y}\sum_{y\in Y}\tilde{g}(x,y)U(y)\le C+\frac{C\beta}{(1-\beta)}=\frac{C}{(1-\beta)},$$

which proves equation (3.5). Taking $U(x)$ to be $\delta_y(x)$, we have $W(x)=g(x,y)$ and $\sup_x W(x)=g(y,y)$. Hence equation (3.6) follows from equation (3.8). Taking $x=y$, we get equation (3.7). □

Remark. If $a(x,y)$ is local, then we can do better if we define $\bar{E}=\bigcup_{x\in E}\{y:a(x,y)>0\}$ and take $C=\sup_{x\in\bar{E}}W(x)$.

We can combine Lemmas 3.1 and 3.2 to obtain the next lemma.

LEMMA 3.4 *For any function u*

$$\sum_{x\in Y}A(x)u^2(x)\le\frac{\beta}{(1-\beta)}\tilde{\mathcal{D}}(u).$$

PROOF: According to Lemma 3.1

$$\sum_{x\in Y}A(x)u^2(x)\le C\tilde{\mathcal{D}}(u)$$

where the constant can be taken to be

$$C=\sup_{x\in Y}\sum_{y\in Y}\tilde{g}(x,y)A(y).$$

In Lemma 3.2 we established that for $x\in Y$

$$\sum_{y\in Y}\tilde{g}(x,y)A(y)\le\frac{\theta(x)}{(1-\beta)}.$$

Taking the supremum over x, clearly we can take C to be $\beta/(1-\beta)$. □

The following results concerning two random walks will be useful:

LEMMA 3.5 *For the symmetric random walk on* $\{\mathbb{Z}^d\setminus\{0\}\}\times\{\mathbb{Z}^d\setminus\{0\}\}$ *with generator*

$$\sum_z a(z)[f(x_1+z,x_2)-f(x_1,x_2)]+\sum_z a(z)[f(x_1,x_2+z)-f(x_1,x_2)],$$

the probability $\theta(x_1,x_2)$ *of hitting the excluded set* $E_2=\{x_1=0\}\cup\{x_2=0\}\cup\{x_1=x_2\}$ *has the property*

$$\sup_{(x_1,x_2)\in(\mathbb{Z}^d\times\mathbb{Z}^d)\setminus E_2}\theta(x_1,x_2):=\alpha<1.$$

PROOF: If we denote by $\delta(x)$ the probability that the continuous time random walk in $\mathbb{Z}^d$ hits 0 for some $t\ge 0$, then $\theta(x_1,x_2)$ can be bounded by

$$\theta(x_1,x_2)\le\delta(x_1)+\delta(x_2)+\delta(x_1-x_2).$$

We know that $\sup_{x\neq 0}\delta(x) < 1$ and $\delta(x) \to 0$ as $x \to \infty$. This is enough to conclude that $\theta(x_1,x_2)$ stays away from 1 near ∞, and since it is strictly less than 1 for each (x_1,x_2), we are done. $\square$

The number α is a constant of the underlying random walk and $0 < \alpha < 1$. The same α is also a bound for the hitting probability of 0 for a single random walk on $\mathbb{Z}^d$. From Lemma 3.3 we get for the random walk on $\mathbb{Z}^d$ that excludes the origin the following bound for the Green's function:

$$g_0(x,x) \le \frac{1}{(1-\alpha)} g(x,x) = \frac{g(0,0)}{(1-\alpha)}.$$

We now consider the symmetric simple exclusion processes on $\mathbb{Z}^d \setminus \{0\}$. The state space Ω consists of maps η from the countable set $\mathbb{Z}^d \setminus \{0\}$ into $\{0,1\}$. Fix $0 < \rho < 1$ and denote the Bernoulli product measure with density ρ by P_ρ. For $x \in \mathbb{Z}^d$, we define

$$\xi_x(\eta) = \frac{\eta(x)-\rho}{\sqrt{\rho(1-\rho)}}. \tag{3.9}$$

For $A \subset \mathbb{Z}^d \setminus \{0\}$ we define

$$\begin{cases} \xi_A(\eta) = \prod_{x\in A} \xi_x(\eta) & \text{if } A \text{ is nonempty} \\ \xi_A(\eta) = 1 & \text{if } A \text{ is empty.} \end{cases} \tag{3.10}$$

Then $\{\xi_A(\cdot)\}$ is an orthonormal basis for $\mathbf{H} = L_2(\Omega)$. It comes naturally graded as $\mathbf{H} = \bigoplus_{n\ge 0} H_n$ where H_n is the span of ξ_A over sets of cardinality n.

If we write

$$u = \sum_A \tilde{u}(A)\xi_A,$$

then the Dirichlet form can be calculated explicitly, and we obtain

$$D^{\text{ex}}(u) = \frac{1}{2}\sum_{n\ge 1}\sum_{|A|=n}\sum_{x,y} a(x,y)[\tilde{u}(A^{x,y}) - \tilde{u}(A)]^2 := D^{\text{ex}}(\tilde{u})$$

where

$$A^{x,y} = \begin{cases} A & \text{if either } x,y \in A \text{ or } x,y \notin A \\ (A\setminus x)\cup y & \text{if } x \in A \text{ and } y \notin A \\ (A\setminus y)\cup x & \text{if } x \notin A \text{ and } y \in A. \end{cases} \tag{3.11}$$

The computation depends on the simple observation that, for all x,y in X, $\xi_A(\eta^{x,y}) = \xi_{A^{x,y}}(\eta)$.

Let us denote by $\mathcal{X}$ the space of all finite subsets of $\mathbb{Z}^d \setminus \{0\}$, and we write $\mathcal{X}$ as the natural union $\bigcup \mathcal{X}_n$ of spaces of subsets of cardinality n. We will consider functions on $\mathcal{X}$ or $\mathcal{X}_n$, and we assume initially that they are zero outside a finite set of points in $\mathcal{X}$ or $\mathcal{X}_n$. Then $\tilde{u}$ is a function on $\mathcal{X}$, and we can write it as $\tilde{u} = \sum_n \tilde{u}_n$ with $\tilde{u}_n$ being the restriction of $\tilde{u}$ to $\mathcal{X}_n$. Clearly,

$$D^{\mathrm{ex}}(\tilde{u}) = \sum_{n\geq 1} D_n^{\mathrm{ex}}(\tilde{u}_n),$$

where for $f : \mathcal{X}_n \to R$

$$\begin{aligned} D_n^{\mathrm{ex}}(f) &= \frac{1}{2}\sum_{A\in\mathcal{X}_n}\sum_{x,y} a(x,y)[f(A^{x,y}) - f(A)]^2 \\ &= \sum_{A\in\mathcal{X}_n}\sum_{\substack{x\in A\\ y\notin A}} a(x,y)[f((A\setminus x)\cup y) - f(A)]^2 . \end{aligned}$$

The Dirichlet form D^{sh} associated with the shift operator L^{sh} is not graded naturally according to our decomposition. A related Dirichlet form allowing the tagged particle to jump to an occupied site,

$$\hat{D}^{\mathrm{sh}}(u) = \frac{1}{2}\int \sum_x a(x)[u(\tau_x\eta) - u(\eta)]^2\, dP_\rho, \tag{3.12}$$

is naturally graded with

$$\hat{D}_n^{\mathrm{sh}}(u) = \hat{D}_n^{\mathrm{sh}}(\tilde{u}_n) = \frac{1}{2}\sum_{x,|A|=n} a(x)[\tilde{u}_n(\tau_x A) - \tilde{u}_n(A)]^2 .$$

We shall prove later on an estimate of $\hat{D}^{\mathrm{sh}}$ in terms of D^{sh} and D^{ex}.

We need the following concept later on: Suppose $n > m$ and we are given a function g on $\mathcal{X}_n$. The function f on $\mathcal{X}_m$ is defined by $f \geq 0$ and

$$f^2(B) = \sum_{\substack{A:A\supset B\\ |A|=n}} g^2(A).$$

Notice that if we consider g as a wave function, then f^2 is simply the m point function associated with the wave function. A very important relation between f and g is the connection between their kinetic energies, especially the special case $m = 1$, which leads to the semiclassical limit for the kinetic energy of Bose gases. Notice that in our setting there is an interaction between particles, namely, the exclusion rule. Nevertheless, it is still valid.

LEMMA 3.6 *For any $n > m$*

$$D_m^{\mathrm{ex}}(f) \leq \binom{n-1}{m-1} D_n^{\mathrm{ex}}(g).$$

In particular, for $m = 1$ we have for the function

$$f^2(x) = \sum_{A:A\ni x} g^2(A),$$

that

$$D_1^{\mathrm{ex}}(f) \leq D_n^{\mathrm{ex}}(g)\,.$$

PROOF: If B_1 and B_2 are two subsets of size m that differ by one point, namely, $B_2 = B_1 \setminus x \cup y$, then one has the following obvious estimate

$$|f(B_1) - f(B_2)|^2 \leq \sum_{\substack{A:|A|=n \\ A \supset B_1 \\ x \in A,\, y \notin A}} |g(A \setminus x \cup y) - g(A)|^2\,.$$

One multiplies this inequality by $a(x,y)$ and sums over everything in sight. One has to be careful and count the number of times a term like $|g(A \setminus x \cup y) - g(A)|^2$ occurs in the summation on the right-hand side. That number is clearly the number of different subsets of A of size m that include a given $x \in A$, and this yields the combinatorial prefactor $\binom{n-1}{m-1}$. □

The following lemma is the main result of this section:

LEMMA 3.7 *Let $n \geq 2$. For any set A in $\mathcal{X}_n$, i.e., a subset of $\mathbb{Z}^d \setminus \{0\}$ of cardinality n, that consists of n distinct points $x_1, x_2, \ldots, x_n$, we define*

$$W_1(A) = \sum_i a(x_i) \quad \textit{and} \quad W_2(A) = \sum_{i \neq j} a(x_i - x_j)\,.$$

Then for any u defined on $\mathcal{X}_n$ that is in the Dirichlet space,

$$\sum_A W_1(A) u^2(A) \leq \frac{\alpha}{(1-\alpha)} D_n^{\mathrm{ex}}(u)$$

and

$$\sum_A [W_1(A) + W_2(A)] u^2(A) \leq \frac{\alpha}{(1-\alpha)} n D_n^{\mathrm{ex}}(u)\,.$$

PROOF: Define v on $\mathcal{X}_1 = \mathbb{Z}^d \setminus 0$ by

$$v^2(x) = \sum_{A:A \ni x} u^2(A)\,.$$

By Lemma 3.6

$$D_1^{\mathrm{ex}}(v) \leq D_n^{\mathrm{ex}}(u)\,.$$

By definition of v,

$$\sum_{|A|=n} W_1(A) u^2(A) = \sum_{x \neq 0} a(x) v^2(x)\,.$$

By Lemma 3.4 and the remark at the end of Lemma 3.6,

$$\sum_{x \neq 0} a(x) v^2(x) \leq \frac{\alpha}{(1-\alpha)} D_n^{\mathrm{ex}}(v) = \frac{\alpha}{(1-\alpha)} D_n^{\mathrm{ex}}(u)\,.$$

This proves the estimate involving W_1.

To prove the second inequality, define the function w on $\mathcal{X}_2$ by

$$w^2(B) = \sum_{A:A\supset B} u^2(A)\,.$$

We can view w as a symmetric function on $(\mathbb{Z}^d \times \mathbb{Z}^d)\setminus E_2$ and compute its Dirichlet form,

$$\begin{aligned} D_2^{\mathrm{ex}}(w) = \frac{1}{2} &\sum_{\substack{x_1,x_2,y_1\in\mathbb{Z}^d\setminus\{0\}\\ x_1\neq y_1}} a(y_1-x_1)[w(y_1,x_2)-w(x_1,x_2)]^2 \\ &+ \frac{1}{2}\sum_{\substack{x_1,x_2,y_2\in\mathbb{Z}^d\setminus\{0\}\\ x_2\neq y_2}} a(y_2-x_2)[w(x_1,y_2)-w(x_1,x_2)]^2\,; \end{aligned}$$

here the factor $\frac{1}{2}$ appears because each set B is counted twice. Again by Lemma 3.6

$$D_2^{\mathrm{ex}}(w) \le (n-1)D_n^{\mathrm{ex}}(u)\,. \tag{3.13}$$

Clearly,

$$\sum_A [W_1(A)+W_2(A)]u^2(A) = \frac{1}{2}\sum_{\substack{x_1,x_2\in\mathbb{Z}^d\setminus\{0\}\\ x_1\neq x_2}} H(x_1,x_2)w^2(x_1,x_2)$$

where

$$H(x_1,x_2) = a(x_1)+a(x_2)+2a(x_1-x_2)$$

is the sum of all the transition rates into the excluded set for the two random walks.

An easy calculation using Lemma 3.5 establishes

$$\sum_B H(B)w^2(B) = \frac{1}{2}\sum_{x_1\neq x_2} H(x_1,x_2)w^2(x_1,x_2) \le \frac{1}{2}\frac{\alpha}{(1-\alpha)}2D_2^{\mathrm{ex}}(w)\,.$$

This fact together with equation (3.13) proves the lemma. $\square$

Another important Markovian evolution, referred to as Glauber dynamics, will be useful later on. We now collect some of its properties here. The Glauber dynamics has each site flipping back and forth between 0 and 1 as a two-state Markov process. The flip rates from $1\to 0$ and $0\to 1$ are taken to be $1/\rho$ and $1/(1-\rho)$, respectively, so that the generator for a single site is

$$(Gv)(0) = \frac{1}{(1-\rho)}(v(1)-v(0))\,,\quad (Gv)(1) = \frac{1}{\rho}(v(0)-v(1))\,,$$

with the corresponding Dirichlet form, $[v(0)-v(1)]^2$. For the full process on Ω, we need to define the flip operator σ_x at site x,

$$\begin{cases} (\sigma_x\eta)(y) = \eta(y) & \text{if } x\neq y \\ (\sigma_x\eta)(x) = 1-\eta(x) & \text{for } x\in X\,. \end{cases}$$

The Glauber generator then has the Dirichlet form

$$D_g(u) = \sum_x D_{g,x}(u)$$

where

$$D_{g,x}(u) = \int_{\Omega} [u(\sigma_x \eta) - u(\eta)]^2 dP_\rho \,. \tag{3.14}$$

Our next lemma is an estimate of $D_{g,x}(u)$ in terms of $D^{\text{ex}}(u)$ in the transient case. The transience is important because in the Glauber dynamics "particles" are not conserved, whereas in exclusion processes they are conserved. The only way to kill a particle in the exclusion model is to send it to infinity, and transience plays an important role in that.

LEMMA 3.8 *Assume transience. Let $u : \Omega \to \mathbb{R}$ depend on a finite set of coordinates. Then for any $x \in X$,*

$$\int_{\eta(x)=1} [u(\sigma_x \eta) - u(\eta)]^2 dP_\rho \le \frac{1}{(1-\rho)} g(x,x) D^{\text{ex}}(u)$$

and

$$\int_{\eta(x)=0} [u(\sigma_x \eta) - u(\eta)]^2 dP_\rho \le \frac{1}{\rho} g(x,x) D^{\text{ex}}(u) \,.$$

By adding the two inequalities,

$$D_{g,x}(u) \le \frac{1}{\rho(1-\rho)} g(x,x) D^{\text{ex}}(u) \,.$$

PROOF: An elementary calculation shows that for a function

$$u(\eta) = \sum_A \tilde{u}(A) \xi_A(\eta)$$

we have

$$\int_{\eta(x)=1} [u(\sigma_x \eta) - u(\eta)]^2 dP_\rho = \frac{1}{(1-\rho)} \sum_{A: A \ni x} \tilde{u}^2(A) \,,$$

$$\int_{\eta(x)=0} [u(\sigma_x \eta) - u(\eta)]^2 dP_\rho = \frac{1}{\rho} \sum_{A: A \ni x} \tilde{u}^2(A) \,.$$

Adding the two equalities, we get

$$D_{g,x}(u) = \frac{1}{\rho(1-\rho)} \sum_{A: A \ni x} \tilde{u}^2(A) \,.$$

If we define $\tilde{u}_n$ to be the restriction of $\tilde{u}$ to $\mathcal{X}_n$,

$$\sum_{A:A\ni x} \tilde{u}^2(A) = \sum_n v_n^2(x) \quad \text{where} \quad v_n^2(x) = \sum_{\substack{A:|A|=n \\ A\ni x}} \tilde{u}_n^2(A)\,.$$

From Lemma 3.1 $v_n^2(x) \le g(x,x)D_1(v_n)$, and by Lemma 3.6 $D_1^{\mathrm{ex}}(v_n) \le D_n^{\mathrm{ex}}(\tilde{u}_n)$. The lemma follows. □

As corollaries of previous lemmas, we prove Lemma 2.1.

PROOF OF LEMMA 2.1: Clearly, to get a bound of the form

$$|\langle f,v\rangle| \le C_f \sqrt{\sum_{x\in F} D_{g,x}(v)}\,,$$

where f depends only on the coordinates x from F, is just finite-dimensional matrix algebra. We can now use Lemma 3.8 to complete the proof. □

Next we obtain an estimate of $\hat{D}^{\mathrm{sh}}$. We already know that $D^{\mathrm{sh}}(u) \le \hat{D}^{\mathrm{sh}}(u)$, since the $\hat{D}^{\mathrm{sh}}$ form is stronger than the D^{sh} form because it corresponds to the tagged particle exchanging places with an untagged particle, which is not allowed. However, we could make the particle at x disappear, let the tagged particle jump to x, which is now empty, and have our old disappeared particle reappear at $-x$ and accomplish our goal. This means that the $\hat{D}^{\mathrm{sh}}$ form can be estimated in terms of the D^{sh} form and the Glauber forms at x and $-x$. More precisely,

$$\begin{aligned}
&\int_{\eta(x)=1} [u(\tau_x\eta)-u(\eta)]^2 dP_\rho \\
&\quad \le 3\int_{\eta(x)=1} \Big[[u(\sigma_{-x}\tau_x\sigma_x\eta)-u(\tau_x\sigma_x\eta)]^2 + [u(\tau_x\sigma_x\eta)-u(\sigma_x\eta)]^2 \\
&\qquad\qquad + [u(\sigma_x\eta)-u(\eta)]^2\Big] dP_\rho \\
&\quad = 3\frac{\rho}{(1-\rho)} \int_{\eta(-x)=0} [u(\sigma_{-x}\eta)-u(\eta)]^2 + 3\frac{\rho}{(1-\rho)} \int_{\eta(x)=0} [u(\tau_x\eta)-u(\eta)]^2 dP_\rho \\
&\qquad + 3\int_{\eta(x)=1} [u(\sigma_x\eta)-u(\eta)]^2 dP_\rho\,.
\end{aligned}$$

Note that the maps σ_x are not measure preserving when $\rho \ne \frac{1}{2}$, and we therefore pick up the factor $\rho/(1-\rho)$. We can now use Lemma 3.8 to conclude that for some constant C not depending on ρ

$$\hat{D}^{\mathrm{sh}}(u) \le \frac{C}{1-\rho}[D^{\mathrm{ex}}(u) + D^{\mathrm{sh}}(u)]\,. \tag{3.15}$$

4 Some Estimates on the Generator

Recall the generator $L = L^{\mathrm{ex}} + L^{\mathrm{sh}}$ of the environment process defined in equations (1.3) and (1.4). Note also that $L_2 = \mathbf{H} = \bigoplus_{n\geq 0} H_n$ where H_n is spanned by the orthonormal basis $\{\xi_A : |A| = n\}$ with ξ_A defined in equation (3.10). Notice that L is a bounded operator on each H_n and maps it into $H_{n-1} \oplus H_n \oplus H_{n+1}$. There are thus bounded operators $B_{n,n-1} : H_n \to H_{n-1}$, $B_{n,n} : H_n \to H_n$, and $B_{n,n+1} : H_n \to H_{n+1}$ such that for $u \in H_n$

$$Lu = B_{n,n-1}u + B_{n,n}u + B_{n,n+1}u\,.$$

To compute these operators explicitly, we start with the action of L^{ex} on ξ_A,

$$L^{\mathrm{ex}}\xi_A = \sum_{x,y\neq 0} p(y-x)\eta(x)(1-\eta(y))[\xi_{A^{x,y}} - \xi_A]\,.$$

Recall $p(x) = a(x) + b(x)$, the sum of its symmetric and asymmetric parts. Hence we obtain

$$L^{\mathrm{ex}}\xi_A = \frac{1}{2}\sum_{x,y\neq 0} a(y-x)[\xi_{A^{x,y}} - \xi_A] + \frac{1}{2}\sum_{x,y\neq 0} b(y-x)(\eta(x)-\eta(y))[\xi_{A^{x,y}} - \xi_A]\,.$$

To proceed further, we write

$$\eta(x) - \eta(y) = \sqrt{\rho(1-\rho)}[\xi_x - \xi_y]$$

and use the rule

$${\xi_x}^2 = 1 + \frac{1-2\rho}{\sqrt{\rho(1-\rho)}}\xi_x$$

to obtain

$$[\eta(x)-\eta(y)][\xi_{A^{x,y}} - \xi_A] = \begin{cases} 2\sqrt{\rho(1-\rho)}[\xi_{A\cup y} - \xi_{A\setminus x}] + \frac{2\rho-1}{\sqrt{\rho(1-\rho)}}[\xi_A + \xi_{A\setminus x\cup y}] & \text{if } x\in A, y\notin A,\\ 2\sqrt{\rho(1-\rho)}[\xi_{A\setminus y} - \xi_{A\cup x}] - \frac{2\rho-1}{\sqrt{\rho(1-\rho)}}[\xi_A + \xi_{A\setminus y\cup x}] & \text{if } x\notin A, y\in A,\\ 0 & \text{in all other cases.}\end{cases}$$

This yields

$$
\begin{aligned}
L^{\text{ex}}\xi_A = {} & \frac{1}{2}\sum_{\substack{x,y\neq 0\\ x\in A, y\notin A}} a(y-x)[\xi_{A\setminus x\cup y}-\xi_A] + \frac{1}{2}\sum_{\substack{x,y\neq 0\\ x\notin A, y\in A}} a(y-x)[\xi_{A\setminus y\cup x}-\xi_A] \\
& + \frac{2\rho-1}{2}\sum_{\substack{x,y\neq 0\\ x\in A, y\notin A}} b(y-x)[\xi_{A\setminus x\cup y}+\xi_A] \\
& - \frac{2\rho-1}{2}\sum_{\substack{x,y\neq 0\\ x\notin A, y\in A}} b(y-x)[\xi_{A\setminus y\cup x}+\xi_A] \\
& + \sqrt{\rho(1-\rho)}\sum_{\substack{x,y\neq 0\\ x\in A, y\notin A}} b(y-x)\xi_{A\cup y} - \sqrt{\rho(1-\rho)}\sum_{\substack{x,y\neq 0\\ x\notin A, y\in A}} b(y-x)\xi_{A\cup x} \\
& - \sqrt{\rho(1-\rho)}\sum_{\substack{x,y\neq 0\\ x\in A, y\notin A}} b(y-x)\xi_{A\setminus x} + \sqrt{\rho(1-\rho)}\sum_{\substack{x,y\neq 0\\ x\notin A, y\in A}} b(y-x)\xi_{A\setminus y}\,.
\end{aligned} \tag{4.1}
$$

We have arranged the terms in equation (4.1) so that the first line is a symmetric piece of $B_{n,n}$, the second two are an asymmetric piece again of $B_{n,n}$, and the last two lines are parts of $B_{n,n+1}$ and $B_{n,n-1}$. In fact, there is a symmetry relative to the interchange of x and y and the first and second terms on each line are equal. We use this to rewrite

$$
\begin{aligned}
& L^{\text{ex}}\xi_A = \\
& \sum_{\substack{x,y\neq 0\\ x\in A, y\notin A}} a(y-x)[\xi_{A\setminus x\cup y}-\xi_A] + (2\rho-1)\sum_{\substack{x,y\neq 0\\ x\in A, y\notin A}} b(y-x)[\xi_{A\setminus x\cup y}+\xi_A] \\
& + 2\sqrt{\rho(1-\rho)}\sum_{\substack{x,y\neq 0\\ x\in A, y\notin A}} b(y-x)\xi_{A\cup y} - 2\sqrt{\rho(1-\rho)}\sum_{\substack{x,y\neq 0\\ x\in A, y\notin A}} b(y-x)\xi_{A\setminus x}\,.
\end{aligned} \tag{4.2}
$$

Now we compute in a similar fashion the term $L^{\text{sh}}\xi_A$. Although the tagged particle cannot jump to a site where there is already a particle present, it is convenient to extend the definition of τ_x to the set $\{x : \eta(x) = 1\}$ by allowing the tagged particle to jump to the site $-x$, which now becomes the origin, and move the particle that was originally at $-x$ back to where the tagged particle was, which is now x. This defines a version of τ_x with $\tau_x\xi_A = \xi_{\tau_x A}$ where

$$
\tau_x A = \begin{cases} A+x & \text{if } -x\notin A \\ (A+x)\setminus 0\cup x & \text{if } -x\in A\,. \end{cases}
$$

Note that $x \in \tau_x A$ if and only if $-x \in A$. After careful calculation we obtain

$$
\begin{aligned}
L^{\mathrm{sh}}\xi_A = {} & (1-\rho)\sum_{-x\notin A} a(x)[\xi_{\tau_x A} - \xi_A] + \rho \sum_{-x\in A} a(x)[\xi_{\tau_x A} - \xi_A] \\
& + (1-\rho)\sum_{-x\notin A} b(x)[\xi_{\tau_x A} + \xi_A] + \rho \sum_{-x\in A} b(x)[\xi_{\tau_x A} + \xi_A] \\
& + \sqrt{\rho(1-\rho)}\sum_{x\notin A} p(x)\xi_{A\cup x} - \sqrt{\rho(1-\rho)}\sum_{-x\notin A} p(x)\xi_{(A+x)\cup x} \\
& + \sqrt{\rho(1-\rho)}\sum_{x\in A} p(x)\xi_{A\setminus x} - \sqrt{\rho(1-\rho)}\sum_{-x\in A} p(x)\xi_{(A+x)\setminus 0}.
\end{aligned} \tag{4.3}
$$

Again the first line is the symmetric part of $B_{n,n}$, followed by its asymmetric part and then the pieces of $B_{n,n+1}$ and $B_{n,n-1}$, respectively.

Putting together L^{ex} and L^{sh}, we have

$$
\begin{aligned}
& B_{n,n}\xi_A \\
& \quad = \sum_{\substack{x,y\neq 0 \\ x\in A, y\notin A}} a(y-x)[\xi_{A\setminus x\cup y} - \xi_A] + (1-\rho)\sum_{-x\notin A} a(x)[\xi_{\tau_x A} - \xi_A] \\
& \qquad + \rho\sum_{-x\in A} a(x)[\xi_{\tau_x A} - \xi_A] + (2\rho - 1)\sum_{\substack{x,y\neq 0 \\ x\in A, y\notin A}} b(y-x)[\xi_{A\setminus x\cup y} + \xi_A] \\
& \qquad + (1-\rho)\sum_{-x\notin A} b(x)[\xi_{\tau_x A} + \xi_A] + \rho\sum_{-x\in A} b(x)[\xi_{\tau_x A} + \xi_A]
\end{aligned} \tag{4.4}
$$

$$
\begin{aligned}
B_{n,n+1}\xi_A = \sqrt{\rho(1-\rho)}\Bigg[& 2\sum_{\substack{x,y\neq 0 \\ x\in A, y\notin A}} b(y-x)\xi_{A\cup y} \\
& + \sum_{x\notin A} p(x)\xi_{A\cup x} - \sum_{-x\notin A} p(x)\xi_{(A+x)\cup x}\Bigg],
\end{aligned} \tag{4.5}
$$

$$
\begin{aligned}
B_{n+1,n}\xi_A = \sqrt{\rho(1-\rho)}\Bigg[& -2\sum_{\substack{x,y\neq 0 \\ x\in A, y\notin A}} b(y-x)\xi_{A\setminus x} \\
& + \sum_{x\in A} p(x)\xi_{A\setminus x} - \sum_{-x\in A} p(x)\xi_{(A+x)\setminus 0}\Bigg].
\end{aligned} \tag{4.6}
$$

First, we will provide some estimates on $B_{n,n+1}$ and $B_{n+1,n}$. If we separate out the odd and even parts,

$$
B_{n,n+1} = \sqrt{\rho(1-\rho)}[B_{n,n+1}^{\mathrm{odd}} + B_{n,n+1}^{\mathrm{even}}]
$$

where the odd and even parts are given by

$$
\begin{aligned}
B^{\mathrm{odd}}_{n,n+1}(\xi_A) &= \left[2\sum_{\substack{x,y\neq 0\\ x\in A, y\notin A}} b(y-x)\xi_{A\cup y} + \sum_{x\notin A} b(x)\xi_{A\cup x} - \sum_{-x\notin A} b(x)\xi_{(A+x)\cup x}\right],\\
B^{\mathrm{even}}_{n,n+1}(\xi_A) &= \left[\sum_{x\notin A} a(x)\xi_{A\cup x} - \sum_{-x\notin A} a(x)\xi_{(A+x)\cup x}\right].
\end{aligned}
\tag{4.7}
$$

The even part of $B_{n+1,n}$ is the adjoint of the even part of $B_{n,n+1}$, and for the odd part an extra change of sign is involved. So if we bound the odd and even pieces of $B_{n,n+1}$ separately, the dual bounds are valid for $B_{n+1,n}$. Our basic estimate is the following:

LEMMA 4.1 *For all $n \geq 1$ and $u \in H_n$ and $v \in H_{n+1}$,*

$$
|\langle B_{n,n+1}u, v\rangle| \leq C\sqrt{\rho(1-\rho)}\sqrt{D^{\mathrm{ex}}_{n+1}(v)}\left[\sqrt{n}\sqrt{D^{\mathrm{ex}}_n(u)} + \sqrt{\hat{D}^{\mathrm{sh}}_n(u)}\right]
$$

and

$$
|\langle B_{n+1,n}v, u\rangle| \leq C\sqrt{\rho(1-\rho)}\sqrt{D^{\mathrm{ex}}_{n}(u)}\left[\sqrt{n}\sqrt{D^{\mathrm{ex}}_{n+1}(v)} + \sqrt{\hat{D}^{\mathrm{sh}}_{n+1}(v)}\right]
$$

where C is a constant that is independent of n and ρ.

PROOF: We shall only prove the first estimate. The second one follows in a similar way or by invoking duality. Write functions u and v in H_n and H_{n+1}, respectively, with expansions $u = \sum_{|A|=n}\tilde{u}(A)\xi_A$ and $v = \sum_{|D|=n+1}\tilde{v}(D)\xi_D$. From equation (4.7), the estimate for $B^{\mathrm{odd}}_{n,n+1}$ reduces to the estimation of

$$
\begin{aligned}
&\left|2\sum_{|A|=n}\sum_{x\in A, y\notin A} b(y-x)\tilde{u}(A)\tilde{v}(A\cup y)\right|\\
&\quad + \left|\sum_{|A|=n}\sum_{x\notin A} b(x)\tilde{u}(A)\tilde{v}(A\cup x) - \sum_{|A|=n} l \sum_{-x\notin A} b(x)\tilde{u}(A)\tilde{v}((A+x)\cup x))\right|.
\end{aligned}
$$

The first term is estimated by

$$
\begin{aligned}
&\left|2\sum_{|D|=n+1}\sum_{x,y\in D} b(y-x)\tilde{u}(D\setminus y)\tilde{v}(D)\right| \\
&\quad=\left|\sum_{|D|=n+1}\sum_{x,y\in D} b(y-x)(\tilde{u}(D\setminus y)-\tilde{u}(D\setminus x))\tilde{v}(D)\right| \quad (b \text{ is odd}) \\
&\quad\le \sum_{|D|=n+1}\sum_{x,y\in D} a(y-x)|\tilde{u}(D\setminus y)-\tilde{u}(D\setminus x)||\tilde{v}(D)| \quad (\text{because } |b|\le a) \\
&\quad\le \left[\sum_{|D|=n+1}\sum_{x,y\in D} a(y-x)|\tilde{u}(D\setminus y)-\tilde{u}(D\setminus x)|^2\right]^{\frac{1}{2}} \\
&\qquad\times\left[\sum_{|D|=n+1}\sum_{x,y\in D} a(y-x)|\tilde{v}(D)|^2\right]^{\frac{1}{2}}.
\end{aligned} \tag{4.8}
$$

The first expression

$$
\left[\sum_{|D|=n+1}\sum_{x,y\in D} a(y-x)|\tilde{u}(D\setminus y)-\tilde{u}(D\setminus x)|^2\right]
$$

is seen to equal $2D_{ex,n}(u)$. In view of Lemma 3.7, the second expression is bounded by $CnD_{ex,n+1}(v)$.

We turn to the second term. This leads to

$$
\begin{aligned}
&\left|\sum_{\substack{x,D\\ x\in D,|D|=n+1}} b(x)[\tilde{u}(D\setminus x)-\tilde{u}((D\setminus x)-x)]\tilde{v}(D)\right| \\
&\quad\le\left[\sum_{\substack{x,D\\ x\in D,|D|=n+1}} a(x)[\tilde{u}(D\setminus x)-\tilde{u}((D\setminus x)-x)]^2\right]^{\frac{1}{2}}\left[\sum_{\substack{x,D\\ x\in D,|D|=n+1}} a(x)[\tilde{v}(D)]^2\right]^{\frac{1}{2}} \\
&\quad=\left[\sum_{\substack{x,A\\ x\notin A,|A|=n}} a(x)[\tilde{u}(A)-\tilde{u}(A-x)]^2\right]^{\frac{1}{2}}\left[\sum_{\substack{x,D\\ x\in D,|D|=n+1}} a(x)[\tilde{v}(D)]^2\right]^{\frac{1}{2}}.
\end{aligned} \tag{4.9}
$$

We have used the inequality $|b(x)| \le a(x)$ in the last step and used $A = D \setminus x$ as the summation variable. The first expression is estimated by

$$\sum_{\substack{x,A \\ x\notin A, |A|=n}} a(x)[\tilde{u}(A) - \tilde{u}(A-x)]^2 = \sum_{\substack{x,A \\ x\notin A, |A|=n}} a(x)[\tilde{u}(\tau_x A) - \tilde{u}(A)]^2$$
$$\le \sum_{\substack{x,A \\ x\notin A, |A|=n}} a(x)[\tilde{u}(\tau_x A) - \tilde{u}(A)]^2 = 2\hat{D}_n^{\mathrm{sh}}(\tilde{u})$$

and the second by

$$\sum_{\substack{x,D \\ x\in D, |D|=n+1}} a(x)[\tilde{v}(D)]^2 \le \frac{\alpha}{(1-\alpha)} D_{n+1}^{\mathrm{ex}}(\tilde{v}) .$$

The estimate for $B_{n,n+1}^{\mathrm{even}}$ is similar. So we have proved the first estimate of Lemma 4.1. The second one follows from duality. This concludes the proof of the lemma. $\square$

5 Some Estimates on the Resolvent

The main theorem of this section is the following:

THEOREM 5.1 *Let $0 < \rho < 1$ and a local mean zero function f be given. Then the solution u_λ of the resolvent equation* (2.10) *satisfies the following estimates*: *For every $k \ge 0$ there is a constant C_k independent of n and ρ such that*

$$\sup_{\lambda>0} \sum_n n^{2k} [D_n^{\mathrm{ex}}(u_\lambda) + (1-\rho)\hat{D}_n^{\mathrm{sh}}(u_\lambda)] \le C_k C_f , \quad \sup_{\lambda>0} \lambda \sum_n n^{2k} \|u_{\lambda,n}\|_0^2 \le C_k C_f .$$

PROOF: We first recall standard estimates. Multiplying equation (2.10) by u_λ and integrating,

$$\lambda \|u_\lambda\|_0^2 + D^{\mathrm{ex}}(u_\lambda) + D^{\mathrm{sh}}(u_\lambda) = \langle f, u_\lambda \rangle \le C_f \sqrt{D^{\mathrm{ex}}(u_\lambda)} .$$

This leads immediately to the estimates

$$\sup_{\lambda>0} \lambda \|u_\lambda\|_0^2 \le C_f , \tag{5.1}$$

$$\sup_{\lambda>0} D^{\mathrm{ex}}(u_\lambda) \le C_f , \tag{5.2}$$

$$\sup_{\lambda>0} D^{\mathrm{sh}}(u_\lambda) \le C_f . \tag{5.3}$$

Because of estimate (3.15), we also have

$$\sup_{\lambda>0} \hat{D}^{\mathrm{sh}}(u_\lambda) \le \frac{1}{(1-\rho)} C_f . \tag{5.4}$$

Let us define the operator T as multiplication by a scalar $t(n)$ on each H_n. The sequence $t(n)$ is assumed to be positive, increasing, and eventually constant. Since L is bounded on each H_n and T is a multiple of I except on a finite number of H_n, it is easily verified that T leaves the domain of L invariant and the commutator $[T,L] = TL - LT$ is a bounded operator from $\mathbf{H} \to \mathbf{H}$. For $u = \sum_n u_n$ with $u_n \in H_n$, an explicit calculation yields

$$[TL - LT]u = \sum_n \left[(t(n+1) - t(n))B_{n,n+1}u_n + (t(n-1) - t(n))B_{n,n-1}u_n\right]$$

and

$$\begin{aligned}\langle [TL - LT]u, Tu\rangle &= \sum_n t(n+1)(t(n+1) - t(n))\langle B_{n,n+1}u_n, u_{n+1}\rangle \\ &\quad + \sum_n t(n-1)(t(n-1) - t(n))\langle B_{n,n-1}u_n, u_{n-1}\rangle\,. \end{aligned} \tag{5.5}$$

From Lemma 4.1, we have

$$\begin{aligned}|\langle [LT - TL]u, Tu\rangle| \le C &\sum_n t(n+1)|t(n+1) \\ &- t(n)|\sqrt{\rho(1-\rho)}\sqrt{D^{\mathrm{ex}}_{n+1}(u)}\left[\sqrt{n}\sqrt{D^{\mathrm{ex}}_n(u)} + \sqrt{\hat{D}^{s}_n(u)}\right] \\ &+ C\sum_n t(n-1)|t(n-1) \\ &- t(n)|\sqrt{\rho(1-\rho)}\sqrt{D^{\mathrm{ex}}_{n-1}(u)}\left[\sqrt{n}\sqrt{D^{\mathrm{ex}}_n(u)} + \sqrt{\hat{D}^{\mathrm{sh}}_n(u)}\right].\end{aligned}$$

Since $D^{\mathrm{ex}}_n(u) = t(n)^{-2}D^{\mathrm{ex}}_n(Tu)$, the last term is equal to

$$\begin{aligned}C\sum_n &\left|\frac{t(n+1)}{t(n)} - 1\right|\sqrt{\rho(1-\rho)}\sqrt{D^{\mathrm{ex}}_{n+1}(Tu)}\left[\sqrt{n}\sqrt{D^{\mathrm{ex}}_n(Tu)} + \sqrt{\hat{D}^{\mathrm{sh}}_n(Tu)}\right] \\ &+ \sum_n \left|\frac{t(n-1)}{t(n)} - 1\right|\sqrt{\rho(1-\rho)}\sqrt{D^{\mathrm{ex}}_{n-1}(Tu)}\left[\sqrt{n}\sqrt{D^{\mathrm{ex}}_n(Tu)} + \sqrt{\hat{D}^{\mathrm{sh}}_n(Tu)}\right].\end{aligned}$$

We now make the choice of $t(n)$ such that for every n, $t(n)$ satisfies

$$\frac{C(1+\sqrt{n})}{2}\left\{\left|\frac{t(n+1)}{t(n)} - 1\right| + \left|\frac{t(n-1)}{t(n)} - 1\right|\right\} \le \delta \tag{5.6}$$

where δ will be chosen soon. This can be achieved by making $t(n) = n_0^k$ for $1 \le n \le n_0$ and $t(n) = n^k$ for $n_0 \le n \le n_1$ and $t(n) = n_1^k$ for $n \ge n_1$. Whereas the choice of n_0 will be governed by δ and k, n_1 can be totally arbitrary and the estimates will be uniform in n_1. In the end n_1 can be allowed to go to infinity, providing us with estimates for the choice of $t(n) = (n_0 \vee n)^k$.

From the factor $\sqrt{\rho(1-\rho)}$, we keep only $\sqrt{(1-\rho)}$ and use it only with $\hat{D}_n^{\mathrm{sh}}(u)$ terms. Then

$$
\begin{aligned}
&|\langle [LT-TL]u, Tu\rangle| \\
&\leq \delta \sum_n \left[D_{n-1}^{\mathrm{ex}}(Tu) + D_{ex,n}(Tu) + D_{n+1}^{\mathrm{ex}}(Tu) + (1-\rho)\hat{D}_n^{\mathrm{sh}}(Tu)\right] \qquad (5.7)\\
&= \delta[3D^{\mathrm{ex}}(Tu) + (1-\rho)\hat{D}^{\mathrm{sh}}(Tu)] \leq \delta C D(Tu)
\end{aligned}
$$

where the constant C comes from equation (3.15). We pick δ so that $C\delta < \frac{1}{4}$. Let us remark that the estimates depend on T only through δ.

From the resolvent equation we have

$$\lambda T u_\lambda - L T u_\lambda = [T, L]u_\lambda + Tf\,.$$

Multiply both sides by Tu_λ and integrate. From equation (5.7) and Lemma 5.1,

$$\lambda \|Tu_\lambda\|_0^2 + D^{env}(Tu_\lambda) \leq \frac{1}{4} D^{env}(Tu_\lambda) + C_{Tf}\sqrt{D^{\mathrm{ex}}(u_\lambda)}\,.$$

Clearly this is sufficient to give the estimates

$$\sup_\lambda \lambda \|Tu_\lambda\|_0^2 \leq C_{Tf}\,, \quad \sup_\lambda \sum_n [t(n)]^2 [D_n^{\mathrm{ex}}(u_\lambda) + (1-\rho)\hat{D}_n^{\mathrm{sh}}(u)] \leq C_0 C_{Tf}^2\,.$$

Since f is local, C_{Tf} is easily controlled. □

6 Proof of Theorem 1.1

We first rewrite $B_{n,n}$ as follows:

LEMMA 6.1 *The operator $B_{n,n}$ from $H_n \to H_n$ can be rewritten as*

$$
\begin{aligned}
B_{n,n}\xi_A = &\sum_{\substack{x,y\neq 0\\ x\in A, y\notin A}} q(y-x)[\xi_{A\setminus x\cup y} - \xi_A] + (1-\rho)\sum_{-x\notin A} p(x)[\xi_{\tau_x A} - \xi_A] \\
&+ \rho \sum_{-x\in A} p(x)[\xi_{\tau_x A} - \xi_A]
\end{aligned}
$$

where $q(x) = q_\rho(x) = a(x) + (2\rho - 1)b(x) = \rho p(x) + (1-\rho)p(-x)$.

PROOF: We have to compare with the earlier expression and make sure that the difference is zero. For this we need to show that

$$(2\rho - 1)\sum_{\substack{x,y\neq 0\\ x\in A, y\notin A}} b(y-x) + (1-\rho)\sum_{-x\notin A} b(x) + \rho \sum_{-x\in A} b(x) = 0\,.$$

If we use the antisymmetry of $b(\cdot)$ as well as its consequence $\sum_x b(x) = 0$, the above relation is easily seen to be true. □

Our next step is to make $B_{n,n}$ look more like a convolution operator. The full translation symmetry is not available because the set $\mathcal{X}_n$ corresponds only to distinct nonzero n-tuples. A subset $A \subset \mathbb{Z}^d \setminus 0$ of cardinality n is really an equivalence class of $n!$ points in $(\mathbb{Z}^d)^n$. All the functions that we consider on $(\mathbb{Z}^d)^n$ will be symmetric under permutation. Let us denote by $\mathcal{G}_n \subset (\mathbb{Z}^d)^n$ the collection of distinct ordered nonzero n-tuples. A function on $\mathcal{X}_n$ can be considered as a symmetric function on $\mathcal{G}_n$ and then extended to all of $(\mathbb{Z}^d)^n$ by defining it to be 0 on the complement $(\mathbb{Z}^d)^n \setminus \mathcal{G}_n$. We will decompose this complement into three parts:

$$\mathcal{B}_n^1 = \{(x_1, \ldots, x_n) : x_i \neq 0 \text{ for } 1 \leq i \leq n \text{ and } x_i = x_j \text{ for just one pair}\},$$
$$\mathcal{B}_n^2 = \{(x_1, \cdots, x_n) : x_i \neq x_j \text{ for } 1 \leq i \neq j \leq n \text{ and for just one } i,\ x_i = 0\},$$

and

$$\mathcal{B}_n^3 = \mathcal{G}_n^c \setminus (\mathcal{B}_n^1 \cup \mathcal{B}_n^2).$$

We want to replace the operator $B_{n,n}$ acting on functions defined on $\mathcal{G}_n$ by the following operator $C_{n,n}$ of convolution type acting on the space of functions defined on all of $(\mathbb{Z}^d)^n$. Note that $\mathcal{B}_n^1$ and $\mathcal{B}_n^2$ are the boundary of $\mathcal{G}_n$, and transitions are possible from $\mathcal{G}_n$ only into $(\mathcal{G}_n \cup \mathcal{B}_n^1 \cup \mathcal{B}_n^2)$.

$$\begin{aligned}
&(C_{n,n}v)(x_1, \ldots, x_n) \\
&\quad = \sum_{x,j} q(x)[v(x_1, \ldots, x_j + x, \ldots, x_n) - v(x_1, \ldots, x_n)] \\
&\qquad + (1-\rho) \sum_x p(x)[v(x_1 - x, \ldots, x_j - x, \ldots, x_n - x) - v(x_1, \ldots, x_n)].
\end{aligned}$$

A comparison has to be made with $B_{n,n}$. Given a function u on $\mathcal{X}_n$, we can view it as a symmetric function defined on $\mathcal{G}_n$ and extend it to all of $(\mathbb{Z}^d)^n$ as a symmetric function by making it zero outside $\mathcal{G}_n$. We will abuse the notation somewhat and not distinguish between the three versions of the same function on $\mathcal{X}_n$, on $\mathcal{G}_n$, and on $(\mathbb{Z}^d)^n$. Since we will deal only with symmetric functions, it will not matter. We can also extend in the same fashion the function $f = B_{n,n}u$.

$$\begin{aligned}
&(B_{n,n}u)(x_1, \ldots, x_n) \\
&\quad = \sum_j \sum_{(x_1, \ldots, y, \ldots, x_n) \in \mathcal{G}_n} q(y - x_j)[u(x_1, \ldots, y, \ldots, x_n) - u(x_1, \ldots, x_n)] \\
&\qquad + (1-\rho) \sum_{x \notin \{x_1, \ldots, x_n\}} p(x)[u(x_1 - x, \ldots, x_n - x) - u(x_1, \ldots, x_n)] \\
&\qquad + \rho \sum_j p(x_j)[u(x_1 - x_j, \ldots, -x_j, \ldots, x_n - x_j) - u(x_1, \ldots, x_n)].
\end{aligned}$$

We define h by

$$C_{n,n}u = B_{n,n}u + h$$

and try to estimate h in terms of u. First, let us compute h explicitly. On $\mathcal{B}_n^3$ the function h is identically zero. For a point in $\mathcal{B}_n^2$, consisting of n distinct points $x_1,\ldots,x_n$, exactly one of which is zero,

$$h(x_1,\ldots,x_{n-1},0)=\sum_{x_n:(x_1,\ldots,x_n)\in\mathcal{G}_n} q(x_n)u(x_1,\ldots,x_{n-1},x_n) \\ +(1-\rho)\sum_{x\neq x_1,\ldots,x_n} p(x)u(x_1-x,\ldots,x_n-x)\,.$$

On $\mathcal{B}_n^1$ where a typical point is $(x_1,\ldots,x_{n-1},x_{n-1})$ with distinct nonzero $x_1,\ldots,x_{n-1}$,

$$h(x_1,\ldots,x_{n-1},x_{n-1})=2\sum_{x_n:(x_1,\ldots,x_n)\in\mathcal{G}_n} q(x_n-x_{n-1})u(x_1,\ldots,x_{n-1},x_n)\,.$$

Finally on $\mathcal{G}_n$,

$$h(x_1,\ldots,x_{n-1},x_n) \\ =-\Bigg[\sum_{i\neq j}q(x_i-x_j)+\sum_i q(-x_i)]u(x_1,\ldots,x_n)-(1-\rho)[\sum_j p(x_j)]u(x_1,\ldots,x_n) \\ -\rho\sum_j p(x_j)[u(x_1-x_j,\ldots,-x_j,\ldots,x_n-x_j)-u(x_1,\ldots,x_n)\Bigg]\,.$$

There are two new Dirichlet forms. The forms $D_n^{\mathrm{ex}}(\cdot)$ that we already saw on functions defined on $\mathcal{X}_n$ as well as the new Dirichlet forms

$$\bar{D}_n(u)=\frac{1}{2}\sum_{\substack{i,x_i'\\ x_1,\ldots,x_n}} a(x_i-x_i')\,[u(x_1,\ldots,x_i,\ldots x_n)-u(x_1,\ldots,x_i',\ldots x_n)]^2\,,$$

which corresponds to n free random walks, and

$$\bar{D}_n^{\mathrm{sh}}(u)=\frac{1}{2}\sum_{x,x_1,\ldots,x_n} a(x)[u(x_1-x,\ldots,x_n-x)-u(x_1,\ldots,x_n)]^2\,,$$

which corresponds to shifts. We have the following estimates:

LEMMA 6.2 *If u is any symmetric function of $x_1,\ldots,x_n$ on $(\mathbb{Z}^d)^n$*

$$D_n^{\mathrm{ex}}(u)\le\frac{1}{n!}\bar{D}_n(u)\,.$$

If, in addition, $u\equiv 0$ outside $\mathcal{G}_n$, then for some constant C independent of n and u,

$$\frac{1}{n!}\bar{D}_n(u)\le CD_{\mathrm{ex},n}(u)\quad\text{and}\quad\left|\frac{1}{n!}\bar{D}_n^{\mathrm{sh}}(u)-\hat{D}_n^{\mathrm{sh}}(u)\right|\le CD_n^{\mathrm{ex}}(u)\,.$$

PROOF: The first part is obvious. The factorial is just the number of times each term is counted. Clearly

$$\bar{D}_n(u)\le n!D_n^{\mathrm{ex}}(u)+2(n-1)!\sum_{x_1,\ldots,x_n} a(x_n-x_{n-1})u^2(x_1,\ldots,x_n)\,,$$

and by Lemma 3.7

$$\sum_{x_1,\dots,x_n}\sum_{i\neq j} a(x_i-x_j)u^2(x_1,\dots,x_n)\le C\,n\,D_n^{\mathrm{ex}}(u)\,,$$

and now the second part follows. The third part also follows from Lemma 3.7 and is nearly identical:

$$\left|\frac{1}{n!}\bar{D}_n^{\mathrm{sh}}(u)-\hat{D}_n^{\mathrm{sh}}(u)\right|\le C\sum_{A\in\mathcal{X}_n}\left(\sum_{x\in A}a(x)\right)[u(A)]^2,$$

where we have identified A with $(x_1,\dots,x_n)$. □

We now return to the estimation of h. If we define

$$\langle h,w\rangle=\frac{1}{n!}\sum h(x_1,\dots,x_n)\,w(x_1,\dots,x_n)\,,$$

then $|\langle h,w\rangle|$ is less than

$$\begin{aligned}|\langle h,w\rangle| &\le C\left[\frac{1}{n!}\sum_{x_1,\dots x_n}A(x_1,\dots,x_n)u^2(x_1,\dots,x_n)\right]^{\frac12}\\ &\qquad\times\left[\frac{1}{n!}\sum_{x_1,\dots x_n}A(x_1,\dots,x_n)w^2(x_1,\dots,x_n)\right]^{\frac12}\\ &\le Cn[D_n^{\mathrm{ex}}(u)]^{\frac12}\left[\frac{1}{n!}\bar{D}_n(w)\right]^{\frac12}.\end{aligned}\tag{6.1}$$

We have defined

$$A(x_1,\dots,x_n)=\sum_{i\neq j}a(x_i-x_j)+\sum_j a(x_j)\,.$$

We have used the following facts: The symmetric part q is also a, and p and q are dominated by $2a$.

The next lemma is a simple consequence of Fourier analysis.

LEMMA 6.3 *Let u be a symmetric function of n variables satisfying*

$$\lambda u-C_{n,n}u=v$$

where v satisfies the bound

$$\left|\sum_{x_1,\dots,x_n}v(x_1,\dots,x_n)w(x_1,\dots,x_n)\right|\le C\left[\bar{D}_n(w)\right]^{\frac12}.$$

Then λu satisfies the same bound with the same constant,

$$\left|\sum_{x_1,\dots,x_n}\lambda u(x_1,\dots,x_n)w(x_1,\dots,x_n)\right|\le C[\bar{D}_n(w)]^{\frac12}.$$

PROOF: Denoting by $\mathbb{T}_d^n$ the n-fold product of the d-torus and by $\hat{u}(\theta)$ and $\hat{v}(\theta)$ the Fourier transforms of u and v, respectively, we have

$$[\lambda + \Phi(\theta) + i\Psi(\theta)]\hat{u}(\theta) = \hat{v}(\theta)$$

where

$$\Phi(\theta) + i\Psi(\theta) = \sum_j \sum_x q(x)[1 - \cos(\theta_j x) - i\sin(\theta_j x)] \\ + (1-\rho)\sum_x p(x)\left[1 - \cos\left(x\sum_j \theta_j\right) + i\sin\left(x\sum_j \theta_j\right)\right].$$

Φ and Ψ are real, and $\Phi(\theta) \geq 0$. Moreover,

$$\|v\|_{-1}^2 = \int_{\mathbb{T}_d^n} \frac{|\hat{v}(\theta)|^2}{H(\theta)} d\theta \quad \text{where } H(\theta) = \sum_j \sum_x a(x)[1 - \cos(\theta_j x)]$$

and

$$\|\lambda u\|_{-1}^2 = \int_{\mathbb{T}_d^n} \frac{|\lambda \hat{u}(\theta)|^2}{H(\theta)} d\theta = \int_{\mathbb{T}_d^n} \frac{|\hat{v}(\theta)|^2}{H(\theta)} \frac{|\lambda|^2}{|\lambda + \Phi(\theta) + i\Psi(\theta)|^2} d\theta \\ \leq \int_{\mathbb{T}_d^n} \frac{|\hat{v}(\theta)|^2}{H(\theta)} d\theta,$$

which proves the lemma. □

PROOF OF THEOREM 1.1: We are now ready to prove our main theorem. We start with a solution u_λ of (2.10) with a local f. If we decompose and write $u = \sum_n u_n$ and $f = \sum_n f_n$ with u_n and f_n from H_n, following the notation of (4.5) and (4.6),

$$\lambda u_{\lambda,n} - B_{n,n} u_{\lambda,n} = f_n + B_{n-1,n} u_{\lambda,n-1} + B_{n+1,n} u_{\lambda,n+1} = g_n .$$

As before, we rewrite

$$\lambda u_{\lambda,n} - C_{n,n} u_{\lambda,n} = g_n + h_n .$$

From (6.1) and Theorem 5.1, we have the estimate

$$|\langle h_n, w\rangle| \leq C_f C_k n^{-k} \left[\frac{1}{n!}\bar{D}(w)\right]^{\frac{1}{2}} .$$

From Lemmas 4.1 and 6.3 and Theorem 5.1, we have

$$|\langle w, B_{n-1,n} u_{\lambda,n-1} + B_{n+1,n} u_{\lambda,n+1}\rangle| \leq C_f C_k n^{-k} \left[\frac{1}{n!}\bar{D}(w)\right]^{\frac{1}{2}} .$$

If f is local, $f_n = 0$ for large enough n. By Lemma 6.4, we now conclude that for all large n,

$$|\langle \lambda u_{\lambda,n}, w\rangle| \leq C_f\, C_k\, n^{-k} \left[\frac{1}{n!}\bar{D}(w)\right]^{\frac{1}{2}} .$$

For small n where $f_n \neq 0$, we have from the proof of Lemma 2.1 and Lemma 6.3 that

$$|\langle w, f_n\rangle| \leq C_f\sqrt{D_n^e} \leq C_f\left[\frac{1}{n!}\bar{D}(w)\right]^{\frac{1}{2}}$$

and so for small n by Lemma 6.4,

$$|\langle \lambda u_{\lambda,n}, w\rangle| \leq C_f\left[\frac{1}{n!}\bar{D}(w)\right]^{\frac{1}{2}}.$$

Finally, by the use of Lemma 6.3, adding up contributions with a larger constant,

$$\|\lambda u_\lambda\|_{-1} \leq C_f C_k \sum_n n^{-k},$$

and we are done. □

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