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stochastic processes and their applications

Stochastic Processes and their Applications 103 (2003) 169–209

www.elsevier.com/locate/spa

Conditional survival distributions of Brownian trajectories in a one dimensional Poissonian environment☆

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Received 3 September 2000; received in revised form 21 May 2002; accepted 11 July 2002

Abstract

Large time annealed path measure limits for a one-dimensional Brownian motion, with possibly a small drift, moving among "soft" Poissonian traps are considered. Limits with respect to both scaled and unscaled motions are derived. The results in both cases considered here agree with those shown before for the related model with "hard" traps. The proofs follow by generalizing previous techniques which identify a large clearing empty of traps in which typically the Brownian motion is confined. What is understood then, in the cases studied, is that under the annealed measure the soft traps organize to act in effect as their hard counterparts.

MSC: primary 60K35; secondary 60F05

Keywords: Soft; Obstacles; Taboo process; Confinement; Brownian motion

1. Introduction

The goal of this article is to investigate a certain localization behavior of an "annealed" one dimensional Brownian motion moving in a Poissonian potential. Here "annealed" refers to the situation where we continuously average over the Poissonian environment, as opposed to the "quenched" case where the Poissonian environment is fixed initially for the Brownian dynamics.

The model is as follows. Let $\mathbb{R}$ be the real numbers, and let $\mathbb{S}$ be the set of all locally finite point configurations ω on $\mathbb{R}$. That is, $\omega=\{\omega^i\}$ is a sequence of points which

☆ Research partially supported by NSF grant NSF/DMS-9703811.
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0304-4149/03/$ - see front matter
PII: S0304-4149(02)00189-8

satisfies $|\{i : \omega^i \in I\}| < \infty$ for every bounded interval I and $\omega^i \neq \omega^j$ for $i \neq j$. Let also $W(\cdot): \mathbb{R} \to \mathbb{R}_+$ be a non-negative, non-degenerate, bounded, measurable function with compact support on $[-a, a]$ for $a > 0$. Moreover, to avoid some technicalities, we further impose that W is piecewise continuous on $[-a, a]$ and continuous at the origin where $W(0) > 0$. With respect to a configuration $\omega \in \mathbb{S}$, locate around each point ω^i the potential $W(\cdot - \omega^i)$ and form the function $V : \mathbb{R} \times \mathbb{S} \to \mathbb{R}_+$ defined by $V(x, \omega) = \sum_i W(x - \omega^i)$. Let also $\mathbb{P}$ be the Poisson point measure with intensity v on $\mathbb{S}$. In addition, let $X(t)$ for $t \geqslant 0$ be standard Brownian motion on $\mathbb{R}$. Denote by P_x the Wiener law of $\{X(t)$: $t \geqslant 0\}$ starting from $x \in \mathbb{R}$, and let E_x be its expectation.

Consider the time evolution of trajectories governed by the measure

$$Q_t^h(\mathrm{d}w, \mathrm{d}\omega) = \frac{1}{S_t^h} \exp\left\{hX(t) - \int_0^t V(X(s), \omega)\,\mathrm{d}s\right\} P_0(\mathrm{d}w)\mathbb{P}(\mathrm{d}\omega), \tag{1.1}$$

where $S_t^h = \mathbb{E} \otimes E_0[\exp\{hX(t) - \int_0^t V(X(s))\,\mathrm{d}s\}]$ is the normalization. This measure is, in fact, the annealed conditional probability distribution on surviving Brownian paths with drift h up to time t where killing is understood in the Feynman–Kac sense with respect to the function $V(\cdot)$.

Intuitively, the term $\exp\{-\int_0^t V(X(s), \omega)\,\mathrm{d}s\}$ represents a penalty for Brownian motion to pass within a distance a of a Poisson point. In this sense, the individual potentials $W(\cdot - \omega^i)$ are "soft" obstacles. Heuristically, these obstacles generalize the case of "hard" obstacles where $W = \infty \cdot 1_{[-a,a]}$ and the penalty term is the absolute penalty $1_{[T>t]}$ where T is the entrance time into the set $\bigcup_i [\omega^i - a, \omega^i + a]$.

Now observe that although the Brownian traveler under $\mathrm{d}Q_t^h$ is discouraged from entering neighborhoods of the Poisson points, it is in fact encouraged to journey long distances by the drift term $\exp\{hX(t)\}$. So the process experiences conflicting impulses to stay put or to travel. Heuristically, however, it is reasonable to understand that if the drift is small enough that the first impulse wins out and the process under $\mathrm{d}Q_t^h$ would localize. In fact, this is made more precise in the following claim due to Eisele and Lang (1987) and Sznitman (1995a):

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \otimes E_0 \left[\exp\left\{hX(t) - \int_0^t V(X(s), \omega)\,\mathrm{d}s\right\}\right] \begin{cases} = 0, & |h| \leqslant \beta_0(1) \\ > 0, & |h| > \beta_0(1), \end{cases} \tag{1.2}$$

where the threshold $\beta_0(1)$ is the annealed Lyapunov exponent first introduced in Sznitman, 1995a (see also Chapter 5.3 of Sznitman (1999)). Roughly, $\beta_0(1)$ measures how expensive it is for the process under the influence of the potential V to reach a remote location when it can pick its own time to get there. More carefully, it is shown in Sznitman (1995a, b) that $\beta_0(1) \leqslant v$ and, as $|x| \to \infty$,

$$\mathbb{E} \otimes E_0 \left[\exp\left\{-\int_0^{\mathscr{H}_x} V(X(s), \omega)\,\mathrm{d}s\right\}\right] = \exp\{-\beta_0(1)|x|(1 + \mathrm{o}(1))\}, \tag{1.3}$$

where $\mathscr{H}_x$ is the hitting time of x.

Now although for h small, we comprehend that the motion localizes, the exact scale of this localization is not so clear from (1.2) or (1.3). However, proceeding from intuition gained from Sznitman's "method of enlargement of obstacles" ideas (Sznitman, 1992) (see Sznitman (1995b) for a review in the soft obstacle context), Povel (1997) recently proved a large deviation principle for $t^{-1/3}X(t)$. He proves, in fact, for $|h| < \beta_0(1)$, that $t^{-1/3}X(t)$ satisfies under $\mathrm{d}Q_t^h$ a large deviation principle in the scale $t^{1/3}$ with rate function J given by

$$J(y) = I(y) - hy - c(1, v - |h|), \tag{1.4}$$

where

$$I(y) = \begin{cases} c(1,v) & \text{for } 0 \leqslant |y| \leqslant c_0, \\ v|y| + \pi^2/(2y^2) & \text{for } c_0 < |y| < c^1, \\ vc^1 + \pi^2/(2(c^1)^2) + \beta_0(1)(|y| - c^1) & \text{for } c^1 \leqslant |y|, \end{cases}$$

$$c(1, v - |h|) = \inf_{l \geqslant 0}[(v - |h|)l + \pi^2/2l^2] = 3/2[\pi(v - |h|)]^{2/3},$$

$c^1 = (\pi^2/(v - \beta_0(1)))^{1/3}$, and $c_h = (\pi^2/(v - |h|))^{1/3}$ is the value of l taken in the minimization problem above. Note that $c^1 \leqslant \infty$ in general, but when W is "small" enough, it is shown in Povel (1998) that $\beta_0(1) < v$ and so $c^1 < \infty$.

Povel further concludes that the survival probability under Q_t^h obeys

$$\lim_{t\to\infty} t^{-1/3} \log \mathbb{E} \otimes E_0 \left[\exp\left\{hX(t) - \int_0^t V(X(s), \omega)\, \mathrm{d}s\right\}\right] = -c(1, v - |h|). \tag{1.5}$$

Note that when $h = 0$, (1.5) is the well known Donsker–Varadhan result (Donsker and Varadhan, 1975).

These estimates and comments suggest a certain confinement of surviving paths, up to time t, to displacements of order $t^{1/3}$ in the regime $|h| < \beta_0(1)$. In fact, for "hard obstacles", Schmock (1990) and Povel (1995) compute various path-measure limits of surviving Brownian motion under $\mathrm{d}Q_t^h$ as $t \uparrow \infty$ for the cases $h = 0$ and $0 < |h| < v$, respectively. More specifically, they determine the path-measure limits of surviving Brownian motion in two different scalings: In natural scale $X(\cdot)$, and in $t^{1/3}$-scale $X(\cdot t^{2/3})/t^{1/3}$. Distinct limiting measures are obtained which reflect certain "boundary" interactions. We note similar investigations for "hard obstacles" when $h = 0$ are established by Sznitman (1991) in $d = 2$ and Povel (1999) in $d \geqslant 3$. The aim of this article is to investigate similar limits in the "soft obstacle" environment in $d = 1$.

2. Results

We define $\Omega = C([0,\infty), \mathbb{R})$ and associate to Ω the usual metric which induces uniform convergence on bounded intervals and makes Ω a complete separable metric space. With respect to trajectories $\{X(t)\colon t \geqslant 0\}$ on Ω, let $\mathscr{F}_t = \sigma\{X(s)\colon 0 \leqslant r \leqslant t\}$ and $\mathscr{F}_\infty = \sigma\{\bigcup_{t\geqslant 0} \mathscr{F}_t\}$.

For a bounded open interval $I=(l,r)$ with $l<r$, let T_I be the exit time from I. Let also ϕ_1^I and λ_1^I be the principal Dirichlet $-(1/2)(\mathrm{d}/\mathrm{d}x)$ eigenfunction and eigenvalue on the interval I, respectively. In fact,

$$\phi_1^I(x) = \begin{cases} \sqrt{2/|I|}\sin(\pi(x-l)/|I|) & \text{for } x \in I, \\ 0 & \text{otherwise}, \end{cases}$$

$$\lambda_1^I = \pi^2/(2|I|^2). \tag{2.1}$$

Define the taboo process measure P_x^I, for $x \in I$, by its action on sets $B \in \mathscr{F}_u$:

$$P_x^I[B] = \frac{\mathrm{e}^{\lambda_1^I u}}{\phi_1^I(x)} E_x[1_B 1_{[T_I>u]}\phi_1^I(X(u))]. \tag{2.2}$$

This is well defined, as for sets $B \in \mathscr{F}_u \subset \mathscr{F}_v$ with $u \leqslant v$, we have by the Markov property that $E_x[1_B 1_{[T_I>v]}\phi_1^I(X(v))] = E_x[1_B 1_{[T_I>u]} E_{X(u)}[1_{[T_I>v-u]}\phi_1^I(X(v-u))]]$ and $E_{X(u)}[1_{[T_I>v-u]}\phi_1^I(X(v-u))] = \phi_1^I(X(u))\mathrm{e}^{-\lambda_1^I(v-u)}$. Observe also, by eigenfunction expansion, that the taboo measure is the weak limit of path measures $P_x[\,\cdot\,|T_I>t]$, as $t\uparrow\infty$, which puts weight on trajectories not leaving I up to time t (Knight, 1969).

Define now the process measure Q^I which is the mixture with weight $\phi_1^I(x)/\int\phi_1^I$ governing initial starting points x of the taboo law of Brownian motion conditioned never to leave the set $I+x$. More precisely, using translation-invariance, Q^I may be identified by its expectation with respect to $f_u \in \mathscr{F}_u$,

$$E_{Q^I}[f_u] = \frac{\mathrm{e}^{\lambda_1^I u}}{\int \phi_1^I}\int \mathrm{d}x E_x[f_u(X(\cdot)-x)1_{[T_I>u]}\phi_1^I(X(u))].$$

Our main results are that the localization with respect to the averaged "soft" obstacle environment is virtually the same as with respect to "hard" obstacles. The case of localization in natural scale when $h \neq 0$ is left open, however, and some comments are made at the end of the section.

Theorem 2.1. (1) *The process $X(\cdot)$ under Q_t^0, as $t\uparrow\infty$, converges weakly to standard Brownian motion.*

(2) *The process $t^{-1/3}X(\cdot t^{2/3})$ under Q_t^0, as $t\uparrow\infty$, converges weakly to the process governed by $Q^{(-c_0/2,c_0/2)}$.*

These limits are the same as in the "hard obstacle" case proved by Schmock (1990). Also note similar process limits are established by Sznitman (1991) in $d=2$, and Povel (1999) for $d\geqslant 3$.

To state the next result, we observe when I is in form $I=(0,c)$ for $c>0$, it is proved in Proposition 1 of Povel (1995) that $P_x^{(0,c)}$ converges weakly, as $x\downarrow 0$, to a path measure denoted by $P_0^{(0,c)}$. This process governed by $P_0^{(0,c)}$ has the interpretation that it begins at the origin but immediately speeds into the open interval $(0,c)$. (cf. Proposition 3.2 for an analogous result.)

Theorem 2.2. *Let* $0<|h|<\beta_0(1)$. *The process* $t^{-1/3}X(\cdot t^{2/3})$ *under* Q_t^h *converges weakly to the process governed by* $P_0^{(0,c_h)}$.

This is the same result for hard obstacles shown in (Povel, 1995).

The intuition for the above theorems, perhaps, is that, although the Brownian traveler may pass through the "soft" traps, the Poissonian averaging selects optimally only those point configurations which have the effect of "hard" traps. In fact, the first important estimate in the proofs of these results is to show, as in the two dimensional "hard" obstacle situation (Sznitman, 1991), that the optimal point configurations are those with an open interval of length $c_h t^{1/3}$ surrounded by a dense forest of points which act to prevent departure from this clearing space. With this picture established, largely from estimates in (Povel, 1997), the proofs are completed by adapting the semi-group method used in (Sznitman (1991). Advantage is also taken of certain precise "soft" potential eigenvalue estimates, which are available in $d=1$.

The "clearing-forest" picture which emerges is perhaps of independent interest and we state it here (cf. Theorem 1 Povel, 1999).

Proposition 2.1. *For* $|h|<\beta_0(1)$, $\varepsilon>0$ *and* $t\geqslant 1$, *let* $\mathbb{G}_{\varepsilon,h}=\mathbb{G}_{\varepsilon,h,t}$ *be the set of configurations* ω *which admit a unique empty interval* $t^{1/3}\mathbb{I}^h_{\varepsilon,\omega}$ *bounded by Poisson points where*

$$\mathbb{I}^h_{\varepsilon,\omega}\subset\begin{cases}[-c_0-\varepsilon,c_0+\varepsilon] & \text{for } h=0,\\ [-\varepsilon,c_h+\varepsilon] & \text{for } 0<h<\beta_0(1),\\ [-c_h-\varepsilon,\varepsilon] & \text{for } -\beta_0(1)<h<0,\end{cases}$$

with length $|\mathbb{I}^h_{\varepsilon,\omega}|\in[c_h-\varepsilon,c_h+\varepsilon]$. *In the case* $h=0$, *let us also specify that there exists a constant* $\zeta>0$ *such that the second largest empty interval bounded by Poisson points in* $t^{1/3}[-c_0-2\varepsilon,c_0+2\varepsilon]$ *has size less than* $\zeta\varepsilon t^{1/3}$. *Let also* $\mathbb{B}^h_{\varepsilon,\omega}$ *be the open* ε*-neighborhood of* $\mathbb{I}^h_{\varepsilon,\omega}$.

(A) *We have confinement*: *For* $|h|<\beta_0(1)$ *and all small* $\varepsilon>0$ *we have*

$$\lim_{t\to\infty}Q_t^h(\mathbb{G}_{\varepsilon,h}\cap\{T_{t^{1/3}\mathbb{B}^h_{\varepsilon,\omega}}>t\})=1.$$

(B) *Also, the motion exhausts its confinement*: *For* $|h|<\beta_0(t)$, $\varepsilon,\xi>0$ *where* $\xi\varepsilon<c_h/2$, *and* $\omega\in\mathbb{G}_{\varepsilon,h}$, *let* $\mathbb{J}_{\varepsilon,h,\xi}$ *be the set*

$$\mathbb{J}_{\varepsilon,h,\xi}=\{x\in\mathbb{B}^h_{\varepsilon,\omega}:\ \operatorname{dist}(x,\partial\mathbb{B}^h_{\varepsilon,\omega})>\xi\varepsilon\}.$$

Then, *there exists* $\xi_0=\xi_0(v,h,\beta_0(1))>0$ *such that*, *for all* $\varepsilon<c_h/(2\xi_0)$, we have

$$\lim_{t\to\infty}Q_t^h(\mathbb{G}_{\varepsilon,h}\cap\{T_{t^{1/3}\mathbb{J}_{\varepsilon,h,\xi_0}}\leqslant t\})=1.$$

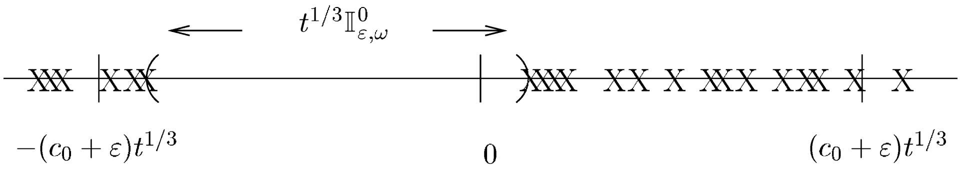

Fig. 1. A possible clearing and forest for $h = 0$.

We remark that, although the errors above are given in terms of a fixed $\varepsilon > 0$, it should certainly be possible to trace through Povel (1997) article and obtain $\varepsilon=\varepsilon(t)=t^{-\gamma}$ for some $\gamma > 0$. However, the γ we could derive in this way would not be large enough to be helpful, say, for the open path-measure problem in natural scale for $h \neq 0$, so we did not pursue this track.

Also, we observe by definition that $\mathbb{G}_{\varepsilon,h}$ already consists of configurations with a unique maximum empty interval and second largest empty interval in $[-2\varepsilon, c_h + 2\varepsilon]$ for $0 < h < \beta_0(1)$ and in $[-c_h - 2\varepsilon, 2\varepsilon]$ for $-\beta_0(1) < h < 0$ on the order $O(\varepsilon t^{1/3})$ as $t \uparrow \infty$. In addition, we note the condition "$\varepsilon\xi < c_h/2$" in part (B) guarantees that $\varepsilon\xi < |\mathbb{B}^h_{\varepsilon,\omega}|/2$ so that $\mathbb{J}_{\varepsilon,h,\xi}$ is well defined (Fig. 1).

We now discuss briefly the unresolved limit in the case of drift $h \neq 0$ in natural scale. For "hard" obstacles, the limit is a mixture of Bessel-3 processes starting from $x \geqslant 0$ with weight proportional to $x\mathrm{e}^{-|h|x}$ Povel (1995). It would be of interest to determine if the limiting statistics for "soft" obstacles are the same, or if the limit measure would allow some initial starting points $x < 0$. The latter situation would have the interpretation that one could begin "outside" the clearing, say, in the forest. To identify this limit, one needs perhaps strict estimates on the drift of the conditioned process outside the clearing; specifically, one must show, as the process ventures into the forest, that the inward drift (into the clearing) becomes strictly positive above the value $\beta_0(1)$ perhaps. This seems technically difficult to prove, though. What is accomplished here, however, to identify the scaled limit, is that the inward drift is positive and increases as the traveler goes into the forest (Lemma 3.10), but more strict bounds are not given.

The plan of the article is as follows. In Section 3, some basic eigenvalue and eigenfunction estimates are stated and developed in three subsections. Section 3.1 gives some general semigroup bounds. In Section 3.2, scaled eigenvalues and eigenfuctions are defined, and estimates in this set-up are made. In Section 3.3, taboo measures which generalize (2.2) in the scaled set-up are considered, and some properties are proved.

In Section 4, we discuss a coarse-graining scheme and prove Proposition 2.1. This discussion is independent of Section 3 except for Section 3.1 and Lemma 3.3 in Section 3.2.

In Section 5, we prove the main theorems. Specifically, in Section 5.1, we prove Theorem 2.1 referring to Proposition 2.1 (A), and Sections 3.1 and 3.2. In Section 5.2, Theorem 2.2 is proved by the scheme of Theorem 2.1, referring to Proposition 2.1 (A), and Proposition 3.2 in Section 3.3.

To clarify expressions, we will occasionally use the notation $I(A) = 1_A$ to denote the indicator function of A and the notation $E[A, B, f, g] = E[I(A)I(B)fg]$ to separate products.

3. Some preliminary estimates

In this section we detail some basic estimates, mostly eigenvalue and eigenfunction estimates, which will be quoted in the next "proofs" section. We begin with a general fact (Section 3.1), then discuss some specific bounds for our Poisson point potential setup (Section 3.2), and then finally give estimates on some taboo measures which figure in the proof of Theorem 2.2 (Section 3.3).

3.1. General estimates

Let $U \in \mathbb{R}$ be a bounded open interval and let $V(x) \geqslant 0$ be a bounded potential on U. Let also T_U be the exit time from U. For $t \geqslant 0$, define $R_t^{U,V}$ as the $L^2(U)$ Feynman–Kac–Wiener semigroup corresponding to the operator $(1/2)(\mathrm{d}^2/\mathrm{d}x^2) - V$ on U given by its action on test functions g:

$$(R_t^{U,V} g)(x) = E_x[g(X(t)), \exp\left(-\int_0^t V(X(s))\,\mathrm{d}s\right), T_U > t].$$

Let $\lambda_i^{U,V}$ for $i = 1, 2, \ldots$ be the increasing sequence of eigenvalues (counted with multiplicity) for the operator $-(1/2)(\mathrm{d}^2/\mathrm{d}x^2) + V$ on U, and $\phi_i^{U,V}$ for $i = 1, 2, \ldots$ be the corresponding orthonormal L^2 eigenfunctions. Note that the principal eigenvalue $\lambda_1^{U,V}$ is simple as the principal eigenfunction $\phi_1^{U,V}$ is unique up to a ± 1 factor. In the following, we will choose $\phi_1^{U,V}$ so that $\phi_1^{U,V} > 0$ is positive on U (cf. Sznitman, 1999, p. 105).

Lemma 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function. For a universal constant C,*

$$\|R_t^{U,V} f\|_{L^\infty} \leqslant C \|f\|_{L^\infty} \sqrt{|U|} \exp\{-\lambda_1^{U,V}(t-1)\},$$

$$\|R_t^{U,V} f - \phi_1^{U,V} \exp\{-\lambda_1^{U,V} t\} < \phi_1^{U,V}, f >_{L^2} \|_{L^2} \leqslant \|f\|_{L^2} \exp\{-\lambda_2^{U,V}\}.$$

Proof. The first statement is proved in Sznitman (1999, Theorem 3.1.2, p. 93). The second follows from standard expansions. □

3.2. Poisson-potential eigen estimates

In order for the general estimates to bear fruit, we need good bounds on the eigenvalues $\lambda_1^{U,V}$ and $\lambda_2^{U,V}$. Also, such bounds will lead to useful eigenfunction estimates. We make these estimates in the Poisson-potential setup.

It will be useful now to generalize the definition of V in the introduction to the set $U\times\mathbb{S}$ for possibly bounded domains $U\subset\mathbb{R}$ by defining

$$V(x,\omega)=\sum_{\omega^i\in U'}W(x-\omega^i),$$

where $U'=\bigcup_{x\in U}\{y\colon |x-y|<a\}$ is the open a-neighborhood of U. When, in particular, U is a bounded interval, the potential $V(x,\omega)$ is bounded on U as there are only a finite number of Poisson points in U'. Moreover, by the regularity assumed on W in the introduction and the previous observation on the bounded number of Poisson points in U', standard results (see, Gilbarg and Trudinger, 1994, Chapter 8) yield that $\phi_1^{U,V}$ is continuously differentiable in U, and continuously piecewise twice differentiable on U.

The following scalings will be important. For $s\geqslant 1$, let $U_s=U/s$ and $\omega_s=\{\omega_s^i\}=\{\omega^i/s\}$ be the s-scaled domain and s-scaled Poisson configuration, respectively. Note that the set of point configurations $\mathbb{S}$ is invariant to the scaling, $\{\omega_s\colon \omega\in\mathbb{S}\}=\mathbb{S}$.

Define also the scaled potential $V_s\colon U_s\times\mathbb{S}\to\mathbb{R}_+$ given by

$$V_s(x,\omega)=s^2V(sx,\omega). \tag{3.1}$$

In the rest of the subsection, we will assume now that U is a bounded open interval.

Lemma 3.2. *For all $s\geqslant 1$ and ω, we have that*

$$\lambda_i^{U_s,V_s}=s^2\lambda_i^{U,V}\quad\text{for}\quad i=1\text{ and }2,\quad\text{and}\quad \phi_1^{U_s,V_s}(x)=s^{1/2}\phi_1^{U,V}(sx). \tag{3.2}$$

Proof. The eigenvalue statement follows from scaling the variational expressions for $\lambda_i^{U,V}$ for $i=1,2$. The first eigenvalue has the formula (Sznitman, 1999, Corollary 1.4.15)

$$\lambda_1^{U,V}=\sup_{\phi}\left\{\int_U(\phi')^2+V\phi^2\,\mathrm{d}x\colon\ \|\phi\|_{L^2}=1\right\} \tag{3.3}$$

for ϕ in the closure of compactly supported smooth functions on U. The statement (3.2) for $i=1$ is actually proved in Lemma 3.1.1 (Sznitman, 1999). Arguments for $i=2$ are similar using the formula for the second eigenvalue: For functions ϕ and ψ in the closure of compactly supported smooth functions on U,

$$\lambda_2^{U,V}=\sup_{\psi}\inf\left\{\int_U(\phi')^2+V\phi^2\,\mathrm{d}x\colon\ \|\phi\|_{L^2}=1,\ \phi\perp\psi\right\}. \tag{3.4}$$

This expression is a corollary of Theorem 1.4.11 (Sznitman, 1999) as Corollary 1.4.15 of (Sznitman, 1999) is for the representation of $\lambda_1^{U,V}$.

Finally, for the eigenfunction statement, it is straightforward to verify, using the scaling relation for the first eigenvalue, that on U_s,

$$\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(s^{1/2}\phi_1^{U,V}(sx))=(\lambda_1^{U_s,V_s}-V_s(x))(s^{1/2}\phi_1^{U,V}(sx)),\quad\text{and}$$

$$\|s^{1/2}\phi_1^{U,V}(s\cdot)\|_{L^2(U_s)}=1.$$

By uniqueness of $\phi_1^{U_s,V_s}$, the eigenfunction part of (3.2) follows. $\square$

It will also be useful to compare the "soft obstacle" eigenfunctions and eigenvalues with their "hard obstacle" counterparts. To this end, observe that the points ω^i of a configuration ω split U into subintervals. Let $|\tau| = |\tau|(U,\omega)$ and $|\tau'| = |\tau'|(U,\omega)$ denote the lengths of the largest and second-largest of these subintervals, with the comment that there may be ties among the subintervals.

However, in the following we will focus mostly on the subset $\mathbb{S}_{1,U} \subset \mathbb{S}$ of configurations ω which possess a unique largest subinterval in U. For $\omega \in \mathbb{S}_{1,U}$, let $\tau = \tau(U,\omega)$ denote this maximal subinterval.

Analogously, in the scaled setup, denote the lengths of the largest and next largest subintervals induced by ω_s in U_s by $|\tau_s| = |\tau_s|(U,\omega) = |\tau|/s$ and $|\tau_s'| = |\tau_s'|(U,\omega) = |\tau'|/s$. Also, for configurations $\omega \in \mathbb{S}_{1,U}$, let $\tau_s = \tau_s(U,\omega) = \tau/s$ denote the largest subinterval induced by ω_s in U_s.

Also, we define the "hard obstacle" eigenvalues and principal eigenfunction. Let

$$\lambda_1^{U,\omega} = \frac{\pi^2}{2|\tau|^2} \quad \text{and} \quad \lambda_2^{U,\omega} = \min\left\{\frac{\pi^2}{2|\tau'|^2}, \frac{2\pi^2}{|\tau|^2}\right\} \tag{3.5}$$

(cf. Sznitman, 1999, Lemma 3.1.1) be the principal and second Dirichlet eigenvalues corresponding to $-(1/2)(\mathrm{d}^2/\mathrm{d}x^2)$ on $U \setminus \omega$. And, for $\omega \in \mathbb{S}_{1,U}$, let $\phi_1^{U,\omega} = \phi_1^{\tau(U,\omega)}$ be the principal L^2 Dirichlet eigenfunction. Note that $\phi_1^{U,\omega}$ is supported on $\tau(U,\omega)$ (cf. (2.1)).

We have the following comparison bounds with respect to the principal Dirichlet eigenvalue and eigenfunction.

Lemma 3.3. *For positive constants $\kappa_1 = \kappa_1(W)$ and $\kappa_2 = \kappa_2(W)$ depending upon the potential W, and for all ω and $s \geqslant 1$, we have that*

$$\min\left\{s^2\kappa_1, \frac{\pi^2}{2(|\tau_s| + \kappa_2 s^{-1})^2}\right\} \leqslant \lambda_1^{U_s,V_s} \leqslant \frac{\pi^2}{2(|\tau_s| - \kappa_2 s^{-1})_+^2}.$$

Proof. The bounds for the unscaled eigenvalues when $s = 1$ are found in Sznitman (1999), (Theorem 3.3.1, p. 123). Then, the bounds for $s \geqslant 1$ follow from the scaling relations in Lemma 3.2. $\square$

Define now, for $r \geqslant 0$, the set $\mathbb{A}_r = \mathbb{A}_r(U) \subset \mathbb{S}_{1,U}$ of configurations ω which have a single large maximum subinterval in U:

$$\mathbb{A}_r(U) = \{\omega\colon r < |\tau| \text{ and } |\tau'| \leqslant |\tau|/10\}.$$

(For what follows it would also be enough to define $\mathbb{A}_r$ so that $|\tau'| \leqslant |\tau|/m$ for an $m > 2$.)

Consider the following notation for ω such that $|\tau|(U,\omega) > 2a$:

$$r_s = |\lambda_1^{U_s,V_s} - \pi^2/(2(|\tau_s| - (2a)s^{-1})^2)|,$$

$$p_s = \lambda_2^{U_s,V_s} - \lambda_1^{U_s,V_s},$$

$$q_s = ((|\tau_s|/(|\tau_s| - (2a)s^{-1}))^2 - 1)/3.$$

Lemma 3.4. *For all $s \geqslant 1$ and all $\omega \in \mathbb{A}_{2a}(U)$, we have that*

$$\|\phi_1^{U_s,V_s} - \phi_1^{U_s,\omega_s}\|_{L^2}^2$$
$$\leqslant 2[r_s/p_s + (1-\sqrt{\max\{1-r_s/p_s,0\}})^2] + 2[q_s + (1-\sqrt{\max\{1-q_s,0\}})^2].$$

Proof. We prove the inequality for $s=1$. Then, the statement for $s \geqslant 1$ will follow by applying scaling relations (3.2), formula $\phi_1^{U_s,\omega_s}(x) = s^{1/2}\phi_1^{U,\omega}(sx)$, and observation that $r_s/p_s = r_1/p_1$ and $q_s = q_1$.

By shifting coordinates, let us assume below that the interval τ takes the form $\tau = (0,|\tau|)$ without loss of generality. To simplify notation, let ϕ be the principal Dirichlet eigenfunction on $(a,|\tau|-a)$ with eigenvalue $\lambda=\pi^2/(2(|\tau|-2a)^2)$. Decomposing ϕ onto the orthonormal basis $\{\phi_i^{U,V}\}$, we have

$$\phi = \sum_{i\geqslant 1}\langle \phi,\phi_i^{U,V}\rangle_{L^2}\phi_i^{U,V}.$$

Also, $\|\phi\|_{L^2}^2 = \sum_{i\geqslant 1}\langle\phi,\phi_i^{U,V}\rangle_{L^2}^2 = 1$, and so

$$1 - \langle\phi,\phi_1^{U,V}\rangle_{L^2}^2 = \sum_{i\geqslant 2}\langle\phi,\phi_i^{U,V}\rangle_{L^2}^2. \tag{3.6}$$

These observations give that

$$\|\phi_1^{U,V} - \phi\|_{L^2}^2 = (1-\langle\phi,\phi_1^{U,V}\rangle_{L^2})^2 + \sum_{i\geqslant 2}\langle\phi,\phi_i^{U,V}\rangle_{L^2}^2$$
$$= (1-\langle\phi,\phi_1^{U,V}\rangle_{L^2})^2 + 1 - \langle\phi,\phi_1^{U,V}\rangle_{L^2}^2.$$

Now, as ϕ is supported on $(a,|\tau|-a)$ where $V=0$, we have $\int_U V\phi^2\,\mathrm{d}x = 0$ and therefore

$$\lambda = \int_U \frac{1}{2}\phi'^2 + V\phi^2\,\mathrm{d}x$$
$$= \sum_{i\geqslant 1}\langle\phi,\phi_i^{U,V}\rangle_{L^2}^2\lambda_i^{U,V}. \tag{3.7}$$

Then as $\lambda_2^{U,V} \leqslant \lambda_i^{U,V}$ for $i \geqslant 2$, we have, inserting $\lambda_1^{U,V} = \lambda_1^{U,V}\sum_{i\geqslant 1}\langle\phi,\phi_i^{U,V}\rangle_{L^2}^2$ into (3.7) and using (3.6), that

$$\lambda - \lambda_1^{U,V} = \sum_{i\geqslant 2}\langle\phi,\phi_i^{U,V}\rangle_{L^2}^2(\lambda_i^{U,V} - \lambda_1^{U,V})$$
$$\geqslant (\lambda_2^{U,V} - \lambda_1^{U,V})\sum_{i\geqslant 2}\langle\phi,\phi_i^{U,V}\rangle_{L^2}^2$$
$$= (\lambda_2^{U,V} - \lambda_1^{U,V})(1-\langle\phi,\phi_1^{U,V}\rangle_{L^2}^2).$$

Therefore, we have that

$$1-\langle\phi,\phi_1^{U,V}\rangle_{L^2}^2 \leqslant r_1/p_1.$$

From the above inequality and near (3.6), we conclude that $1\geqslant\langle\phi,\phi_1^{U,V}\rangle_{L^2}^2\geqslant\max\{1-r_1/p_1,0\}$, and so

$$(\langle\phi,\phi_1^{U,V}\rangle_{L^2}-1)^2\leqslant(1-\sqrt{\max\{1-(r_1/p_1),0\}})^2.$$

All these observations give that $\|\phi-\phi_1^{U,V}\|_{L^2}^2\leqslant r_1/p_1+(1-\sqrt{\max\{1-(r_1/p_1),0\}})^2$.

As the second eigenvalue $\lambda_2^{U,\omega}$ satisfies (3.5) and by assumption $|\tau'|\leqslant|\tau|/10$, we have $\lambda_2^{U,\omega}=2\pi^2/|\tau|^2$. Then, as above, we may decompose ϕ onto $\phi_1^{U,\omega}$ to get that $\|\phi-\phi_1^{U,\omega}\|_{L^2}^2\leqslant q_1+(1-\sqrt{\max\{1-q_1^2,0\}})^2$.

The inequality $(a+b)^2\leqslant 2a^2+2b^2$ completes the proof. □

To apply the last result, we must control the denominator p_s.

Lemma 3.5. *There exist constants $r_0=r_0(W)$ and $\mathscr{C}=\mathscr{C}(W)>1$ such that for all $r\geqslant r_0$, $s\geqslant 1$, and $\omega\in\mathbb{A}_r(U)$ we have that $\lambda_2^{U_s,V_s}\geqslant\mathscr{C}\lambda_1^{U_s,\omega_s}$.*

Proof. We will prove the statement for $s=1$. The result for $s\geqslant 1$ now follows from the scaling relation (Lemma 3.2), and formula $\lambda_1^{U_s,\omega_s}=\pi^2/(2|\tau_s|^2)$.

We will follow the general approach of Lemma 3.3.2 (Sznitman, 1999) which gives estimates for the principal eigenvalue $\lambda_1^{U,V}$. Let $\omega\in\mathbb{S}_{1,U}$ be such that $|\tau'|(U,\omega)\leqslant|\tau|(U,\omega)/10$. To find lower bounds on $\lambda_2^{U,V}$ we work with the formula (3.4). Let $\psi=\phi_1^{U,\omega}$ be the hard obstacle eigenfunction supported in the largest subinterval $\tau(U,\omega)$. Let $\phi\perp\psi$ be given with $\|\phi\|_{L^2}=1$ in the closure of compactly supported smooth functions on U. Let $I_k=I_k(\omega)$ be the subintervals in U marked by the configuration ω. We transform the problem by Dirichlet–Neumann bracketing.

$$\begin{aligned}\int_U\frac12(\phi')^2+V\phi^2\,\mathrm{d}x&=\sum_k\int_{I_k}\frac12(\phi')^2+V\phi^2\,\mathrm{d}x\\&\geqslant\inf_k v(I_k)\sum_k\int_{I_k}\phi^2\,\mathrm{d}x\\&=\inf_k v(I_k)\cdot\int_U\phi^2\,\mathrm{d}x,\end{aligned}$$

where

$$\begin{aligned}v(I_k)=\inf\Bigg\{&\int_0^{|I_k|}\frac12(v')^2+(W(x)+W(x-|I_k|))v^2\,\mathrm{d}x:\\&v\in C^1[0,|I_k|],\int_0^{|I_k|}v^2\,\mathrm{d}x=1\Bigg\}\end{aligned}$$

if $I_k \neq \tau(U,\omega)$. Otherwise, on the largest interval, $v(\tau)$ is as above with the added condition in the infimum that also $v \perp \bar{\psi}$ where $\bar{\psi}$ is ψ shifted to the interval $[0,|\tau|]$ (cf. formulas (3.3) and (3.4)).

With respect to ω, the values of $v(I_k)$ for $I_k \neq \tau$ are the principal eigenvalues on I_k and so are bounded below by $\min\{\kappa_1(W), \pi^2/(2(|\tau|/10 + \kappa_2(W))^2)\}$ from Lemma 3.3.

We now find a lower bound on $v(\tau)$. This is accomplished, by following the optimizations used for Lemma 3.3.2 (Sznitman, 1999). Call $l = |\tau|$ and note the scaling relation: $l^2 v(\tau)$ equals

$$\inf\left\{\int_0^1 \frac{1}{2}(v')^2 + l^2(W(xl) + W(l(x-1)))v^2\,\mathrm{d}x:\right.$$

$$\left. v\in C^1[0,1],\ v\perp\psi_1, \int_0^1 v^2\,\mathrm{d}x = 1\right\},$$

where $\psi_1 = \sin(\pi x)$, the hard obstacle eigenfuction on $[0,1]$. By considering a minimizing sequence in the infimum above we can find a $v_l \in C[0,1]$, $v_l' \in L^2[0,1]$, $\|v_l\|_{L^2} = 1$ where the minimum is taken. Observe that v_l must have a zero in $(0,1)$ due to $v_l \perp \psi_1$. There will now be two cases to consider depending upon a parameter $0 < \delta < 1/4$ to be fixed later. Case 1: There is a zero $x_0 \in (\delta, 1-\delta)$; and Case 2: There is no zero in $(\delta, 1-\delta)$. We will assume in the following that $l > 2a/\delta > 8a$.

Case 1. There is a zero $x_0 \in (\delta, 1-\delta)$.

Let η_+ be the minimum value of $|v_l|$ on the interval $[0, a/l]$ taken at α_l, and η_- be the minimum of $|v_l|$ on $[1-a/l, 1]$ taken at β_l. Let $\eta = \eta_+ + \eta_-$ and define

$$u(x) = v_l(x) - \frac{(x-\alpha_l)(x-x_0)}{(\beta_l - x_0)(\beta_l - \alpha_l)}\,v_l(\beta_l) - \frac{(x-x_0)(\beta_l - x)}{(\alpha_l - x_0)(\beta_l - \alpha_l)}\,v_l(\alpha_l)$$

so that in particular u has zeroes at α_l, β_l and x_0. As $\beta_l - \alpha_l \geqslant 1/2$ and $\beta_l - x_0$, $x_0 - \alpha_l \geqslant \delta/2$, from the assumption on l, we have

$$\|u_l - v_l\|_{L^\infty} \leqslant \frac{2}{(1/2)(\delta/2)}\max(\eta_+, \eta_-), \tag{3.8}$$

and

$$\|u_l' - v_l'\|_{L^\infty} \leqslant \frac{2\cdot 4}{(1/2)(\delta/2)}\max(\eta_+, \eta_-). \tag{3.9}$$

Some computation (cf. Sznitman, 1999, p. 125–126) gives that

$$l^2 v(\tau) \geqslant \frac{1}{2}\max\left\{\frac{\pi}{1-\delta} - \frac{C\eta}{\delta}\left(\frac{\pi}{1-\delta}+1\right), 0\right\}^2 + c\eta^2 l, \tag{3.10}$$

where C is a universal constant and $c=(1/4)\min\{\int_0^a W\,\mathrm{d}x, \int_{-a}^0 W\,\mathrm{d}x\}>0$. Indeed, using (3.9), we have

$$\begin{aligned} l^2 v(\tau) &= \frac{1}{2}\int_0^1 (v_l')^2(x)\,\mathrm{d}x + l^2\int_0^1 (W(xl)+W(l(x-l)))(v_l)^2(x)\,\mathrm{d}x \\ &\geqslant \frac{1}{2}\|v_l'\|_{L^2}^2 + c\eta^2 l \\ &\geqslant \frac{1}{2}\max\left\{\|u_l'\|_{L^2} - \frac{32}{\delta}\eta, 0\right\}^2 + c\eta^2 l. \end{aligned}$$

To estimate further, as $u_l(\alpha_l)=u_l(x_0)=u_l(\beta_l)=0$ and $\max\{\beta_l-x_0, x_0-\alpha_l\}\leqslant 1-\delta$, we have by Poincaré's inequality that

$$\int_{\alpha_l}^{\beta_l} (u_l')^2\,\mathrm{d}x \geqslant \frac{\pi^2}{(1-\delta)^2}\int_{\alpha_l}^{\beta_l} (u_l)^2\,\mathrm{d}x.$$

We now extend u_l to the intervals $[-\alpha_l, 0]$ and $[1, 2-\beta_l]$ by reflection across 0 and 1 so that $u_l(-x)=u_l(x)$ and $u_l(2-y)=u_l(y)$ for $x\in[0,\alpha_l]$ and $y\in[\beta_l,1]$, respectively, (cf. Sznitman, 1999, Figure on p. 126). As $\alpha_l, 1-\beta_l\leqslant 1/8$, we can argue by Poincaré's inequality again that

$$\int_{[0,\alpha_l]\cup[\beta_l,1]} (u_l')^2\,\mathrm{d}x \geqslant 64\pi^2 \int_{[0,\alpha_l]\cup[\beta_l,1]} (u_l)^2\,\mathrm{d}x.$$

Therefore, as $1-\delta\geqslant \frac{3}{4}$, we have $\|u_l'\|_{L^2}\geqslant(\pi^2/(1-\delta)^2)\|u_l\|_{L^2}$. With (3.8) and $\|v_l\|_{L^2}=1$, we then have

$$\begin{aligned} l^2 v(\tau) &\geqslant \frac{1}{2}\max\left\{\frac{\pi}{1-\delta}\|u_l\|_{L^2} - \frac{32}{\delta}\eta, 0\right\}^2 + c\eta^2 l \\ &\geqslant \frac{1}{2}\max\left\{\frac{\pi}{1-\delta} - \left(\frac{8\pi}{\delta(1-\delta)} + \frac{32}{\delta}\right)\eta, 0\right\}^2 + c\eta^2 l \end{aligned}$$

which then implies (3.10) for suitable constants.

Now, minimization of (3.10) on η for l large enough gives

$$l^2 v(\tau) \geqslant \frac{1}{2}\frac{\pi^2}{(1-\delta)^2} - \frac{C(W)}{l}.$$

Case 2: There is no zero in $(\delta, 1-\delta)$.

The guiding intuition nevertheless is that there is a point $y_0\in[\frac{1}{4},\frac{3}{4}]$ where $|v_l(y_0)|$ is small, if not necessarily zero. Consider that $v_l\perp\psi_1$, $\int_0^1 v_l^2\,\mathrm{d}x=1$, $\psi_1\geqslant 0$, and that ψ_1 is increasing on $[0,\frac{1}{2}]$ and symmetric about $\frac{1}{2}$. To make a preliminary estimate, write

$$\left|\int_{\delta}^{1-\delta} v_l\psi_1\,\mathrm{d}x\right| = \left|\int_0^{\delta} v_l\psi_1\,\mathrm{d}x + \int_{1-\delta}^{1} v_l\psi_1\,\mathrm{d}x\right|$$

$$\leqslant 2\left[\int_0^{\delta}\psi_1^2\,\mathrm{d}x\right]^{1/2}\left[\int_0^1 v_l^2\,\mathrm{d}x\right]^{1/2}$$

$$\leqslant 2\psi_1(\delta)\delta^{1/2}.$$

Let $|v|_{\min} = \min_{1/4\leqslant y\leqslant 3/4}|v_l(y)|$, and suppose this minimum is taken at $y_0\in[\frac{1}{4},\frac{3}{4}]$. Then, as by assumption v_l is of one sign on $(\delta, 1-\delta)$, we have

$$|v|_{\min}\int_{1/4}^{3/4}\psi_1(x)\,\mathrm{d}x \leqslant \left|\int_{1/4}^{3/4} v_l(x)\psi_1(x)\,\mathrm{d}x\right|$$

$$\leqslant \left|\int_{\delta}^{1-\delta} v_l(x)\psi_1(x)\,\mathrm{d}x\right|$$

$$\leqslant 2\psi_1(\delta)\delta^{1/2}.$$

This implies the bound $|v|_{\min}\leqslant\sqrt{2}\pi\delta^{3/2}$ from the definition of ψ_1.

Let now $\alpha_l, \beta_l, \eta_+, \eta_-$ and η be as in case 1. Define $u_l(x)$ equal to

$$v_l(x) - \frac{(x-\alpha_l)(x-y_0)}{(\beta_l-y_0)(\beta_l-\alpha_l)}\,v_l(\beta_l) - \frac{(x-y_0)(\beta_l-x)}{(\alpha_l-y_0)(\beta_l-\alpha_l)}\,v_l(\alpha_l)$$

$$-\frac{(x-\alpha_l)(\beta_l-x)}{(y_0-\alpha_l)(\beta_l-y_0)}\,v_l(y_0)$$

and see that u_l vanishes at α_l, y_0 and β_l. From the assumptions on l and δ, we have that $y_0-\alpha_l, \beta_l-y_0\geqslant\frac{1}{8}$ and $\beta_l-\alpha_l\geqslant\frac{1}{2}$. Therefore, we have the bounds

$$\|u_l-v_l\|_{L^\infty}\leqslant 3(8^2)\max\{\eta,|v|_{\min}\}\quad\text{and}\quad\|u_l'-v_l'\|_{L^\infty}\leqslant 6(8^2)\max\{\eta,|v|_{\min}\}.$$

Analogous to case 1, these estimates and Poincaré's inequality applied to u_l (noting $\alpha_l, 1-\beta_l\leqslant\frac{1}{8}$ and $\beta_l-y_0, y_0-\alpha_l\leqslant\frac{3}{4}$) give a lower bound for $l^2\nu(\tau)$ of

$$\frac{1}{2}\max\left\{\frac{4\pi}{3} - C\max\{\eta,|v|_{\min}\}\left(\frac{4\pi}{3}+1\right),0\right\}^2 + c\eta^2 l \tag{3.11}$$

for a universal constant C. When $\eta\leqslant|v|_{\min}$, we have that (3.11) is larger than

$$\frac{1}{2}\left(\frac{4\pi}{3} - \left(\frac{4\pi}{3}+1\right)\sqrt{2}\pi C\delta^{3/2}\right)^2 > \pi^2/2$$

for a small δ that we now fix. In the case $\eta>|v|_{\min}$, we proceed as in case 1 to minimize $(1/2)\max\{4\pi/3 - C\eta(4\pi/3+1),0\}^2 + c\eta^2 l$, over all $\eta\geqslant 0$, to get for l large that (3.11) is bigger than

$$\frac{1}{2}\left(\frac{4\pi}{3}\right)^2 - \frac{C(W)}{l}.$$

Putting together the lower bounds in Cases 1 and 2 with respect to δ and length $l = |\tau|$, yields an $r_0(W) \geqslant 1$ and a constant $\mathscr{C} > 1$ such that $\lambda_2^{U,V} \geqslant \mathscr{C}\lambda_1^{U,\omega}$ for all $\omega \in \mathbb{A}_r$ and $r \geqslant r_0(W)$.

This finishes the proof of the lemma. $\square$

As a byproduct of Lemmas 3.3–3.5, we show that the scaled soft and hard principal eigenfunctions are close in L_∞.

Lemma 3.6. *Let $\alpha, \beta : [1,\infty) \to \mathbb{R}$ be real-valued functions such that $\alpha(s) < \beta(s)$ for all $s \geqslant 1$. Let $U(s) \subset \mathbb{R}$ be the interval $U(s) = (\alpha(s), \beta(s))$ so that $c \leqslant |U(s)_s| \leqslant d$ for $s \geqslant 1$ where $0 < c \leqslant d$. Then, we have*

$$\overline{\lim_{s\to\infty}} \sup_{\omega\in\mathbb{A}_{cs}(U(s))} \|\phi_1^{U(s)_s,V_s} - \phi_1^{U(s)_s,\omega_s}\|_{L^\infty} = 0.$$

Consequently, for all large s,

$$\sup_{\omega\in\mathbb{A}_{cs}(U(s))} \|\phi_1^{U(s)_s,V_s}\|_{L^\infty} \leqslant 2\sqrt{2/c}.$$

Proof. The second statement follows from the first and (2.1) applied to $\phi_1^{U(s)_s,\omega_s} = \phi_1^{\tau_s(U(s),\omega)}$. To simplify notation, let us call $U_s = U(s)_s$. To prove the first statement, without loss of generality, we may assume that $U_s = (0, |U_s|) \subset [0,d]$ for all $s \geqslant 1$, by shifting coordinates. Also, by Lemma 3.3, let $s_0 = s_0(W,c)$ be such that for $s \geqslant s_0$ and $\omega \in \mathbb{A}_{cs}(U(s))$, we have $\lambda_1^{U_s,V_s} \leqslant 2\lambda_1^{U_s,\omega_s}$. Now note that the family

$$\{\phi_1^{U_s,V_s} - \phi_1^{U_s,\omega_s}\colon\ \omega \in \mathbb{A}_{cs}(U(s)),\ s \geqslant s_0\}$$

is uniformly bounded and equicontinuous on $[0,d]$: Clearly at the points $x = 0$ and d the family $\phi_1^{U_s,V_s}(x) - \phi_1^{U_s,\omega_s}(x) \equiv 0$. Also, for $x, y \in [0,d]$,

$$\begin{aligned}
|\phi_1^{U_s,V_s}(x) - \phi_1^{U_s,V_s}(y)| &\leqslant \|(\phi_1^{U_s,V_s})'\|_{L^2}\sqrt{|x-y|}\\
&\leqslant \lambda_1^{U_s,V_s}\sqrt{|x-y|}\\
&\leqslant 2(\pi^2/(2|\tau_s|^2))\sqrt{|x-y|}\\
&\leqslant (\pi^2/c^2)\sqrt{|x-y|}.
\end{aligned}$$

Similarly, $|\phi_1^{U_s,\omega_s}(x) - \phi_1^{U_s,\omega_s}(y)| \leqslant (\pi^2/(2c^2))\sqrt{|x-y|}$.

Let $\{(s',\omega)\colon\ \omega \in \mathbb{A}_{cs'}(U(s')),\ s' \uparrow \infty\}$ be a maximal sequence for the difference in the lemma. Then, for any subsequence of this sequence, there exists a further subsequence $\{(s'',\omega)\}$ such that $\phi_1^{U_{s''},V_{s''}} - \phi_1^{U_{s''},\omega_{s''}}$ converges uniformly to a bounded continuous function ψ on $[0,d]$. From Lemma 3.4, we have that on $\mathbb{A}_{cs''}(U(s''))$, for

$s'' > 2a/c$, that

$$\|\phi_1^{U_{s''},V_{s''}} - \phi_1^{U_{s''},\omega_{s''}}\|_{L^2} \leqslant 2[r_{s''}/p_{s''} + (1-\sqrt{1-\max(r_{s''}/p_{s''},0)})^2] + 2[q_{s''} + (1-\sqrt{1-\max(q_{s''},0)})^2]. \tag{3.12}$$

From Lemmas 3.3 and 3.5, uniformly on $\mathbb{A}_{cs''}(U(s''))$, as $s'' \uparrow \infty$, $r_{s''} = \mathrm{O}((s'')^{-1})$ and $p_{s''} \geqslant \mathscr{C}\lambda_1^{U_{s''},\omega_{s''}} - \lambda_1^{U_{s''},V_{s''}} \geqslant (\mathscr{C}-1)(\pi^2/2c^2) + \mathrm{O}((s'')^{-1})$, and also $q_{s''} = \mathrm{O}((s'')^{-1})$. Then, we have that the right-hand side of (3.12) is $\mathrm{O}((s'')^{-1/2})$ and so $\|\phi_1^{U_{s''},V_{s''}} - \phi_1^{U_{s''},\omega_{s''}}\|_{L^2} \leqslant Cs''^{-1/2}$ for all large s'' with respect to some constant $C = C(W,c)$. Therefore from bounded convergence, $\|\psi\|_{L^2} = 0$, and so, $\psi \equiv 0$, which further implies that $\|\phi_1^{U_{s''},V_{s''}} - \phi_1^{U_{s''},\omega_{s''}}\|_{L^\infty} \to 0$ as $s'' \uparrow \infty$. Hence, by considering subsequences, we finish the proof. □

3.3. Taboo measures

We now give some estimates on some taboo measures which arise in a scaled soft obstacle setting. These bounds will be useful later for the proof of Theorem 2.2.

For $u \geqslant 0$, let f_u be a $\mathscr{F}_u$ measurable function. Recall, for an interval I, the Brownian taboo measures P_x^I defined in (2.2). A key result will be Proposition 1 (Povel, 1995) which we write here for reference.

Proposition 3.1. *As $x \downarrow 0$, $P_x^{(0,c)}$ converges weakly to a probability measure $P_0^{(0,c)}$ on $(\Omega, \mathscr{F}_\infty)$.*

The purpose of this subsection is, in fact, to show that similar results hold in a scaled soft obstacle setup.

For sets $I \subset \mathbb{R}$ and $s \geqslant 1$, define the function $\bar{V}_s : I \times \mathbb{S} \to \mathbb{R}_+$ by

$$\bar{V}_s(x,\omega) = V_s(x, s\omega) \ (= s^2 V(sx, s\omega)), \tag{3.13}$$

where, with respect to the definition of V_s (3.1), the set U takes form $U = sI$.

When I is an open bounded interval, define, with respect to $(x,\omega) \in I \times \mathbb{S}$ and $s \geqslant 1$, a taboo measure $\bar{P}^I_{\omega,x,s}$ on Brownian paths by its expectation with respect to f_u. Namely, $\bar{E}^I_{\omega,x,s}[f_u]$ equals

$$\frac{\exp\{\lambda_1^{I,\bar{V}_s} u\}}{\phi_1^{I,\bar{V}_s}(x)} E_x\left[f_u(X(\cdot)), \exp\left\{-\int_0^u \bar{V}_s(X(r),\omega)\,\mathrm{d}r\right\},\ T_I > u,\right.$$
$$\left.\phi_1^{I,\bar{V}_s}(X(u))\right]. \tag{3.14}$$

As for the Brownian taboo measure in Section 2, this definition is well defined. The taboo measure $\bar{P}^I_{\omega,x,s}$, analogous to the Brownian taboo measures, can also be constructed as the weak limit, as $t \to \infty$, of the conditional distributions,

$$\frac{E_x[\cdot\,, \exp\{-\int_0^t \bar{V}_s(X(r),\omega)\,\mathrm{d}r\}, T_I > t]}{E_x[\exp\{-\int_0^t \bar{V}_s(X(r),\omega)\,\mathrm{d}r\}, T_I > t]}.$$

An eigenfunction expansion gives the formula for $\bar{E}^I_{\omega,x,s}[f_u]$. The taboo measures, in fact, for $x \in I$, form a diffusion process on I with generator

$$\mathscr{L} = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}}{\mathrm{d}x}\log \phi_1^{I,\bar{V}_s}(x)\right)\frac{\mathrm{d}}{\mathrm{d}x} \tag{3.15}$$

such that the left and right boundary points of I are inaccessible.

We now explain the role of $\bar{V}_s$ and some of the intuition for what follows. Later, in the proof of Theorem 2.2, we will consider an interval of the form $I = (-\bar{a}, \bar{b})$ for positive numbers $\bar{a}$ and $\bar{b}$. Also, in the proof, we will restrict ourselves to configurations $\omega \in \mathbb{S}_{1,I}$ where the maximal empty subinterval $\tau \subset I$ is of the form $\tau = (l, r)$ with $|\tau| \sim (\bar{b} + \bar{a})$ and $l \sim -\bar{a}$. With respect to such an ω, the scaled potential $\bar{V}_s(x,\omega) = s^2 \sum_{s\omega^i \in (sI)'} W(s(x - \omega^i))$ as $s \uparrow \infty$ acts more and more like a "hard" potential with tall thin "spikes" of diameter $\sim s^{-1}$ at configuration points ω^i near the boundaries of I. In fact, the heuristic limit as $s \uparrow \infty$ is $\infty \cdot 1_{[x=-\bar{a} \text{ or } \bar{b}]}$. It seems plausible then that the taboo measure $\bar{P}^I_{\omega,x,s}$ on I with respect to configurations ω of this sort might behave like a "hard" taboo measure $P_x^{(-\bar{a},\bar{b})}$ for large s. This is the basic idea which we try to formalize in the main result of this subsection below.

We now specify more carefully the structure of points considered on an interval $I = (-\bar{a}, \bar{b})$. For $\omega \in \mathbb{S}_{1,I}$, let l be the left end-point of $\tau(I,\omega)$ so that $\tau = (l, l + |\tau|)$. In terms of positive parameters $\bar{a}, \bar{b}, \bar{c}$ and β, define configurations $\bar{\mathbb{A}} \subset \mathbb{S}_{1,I}$ by

$$\bar{\mathbb{A}} = \{\omega \in \mathbb{S}_{1,I}\colon\ |l| \leqslant \bar{a}/2 \text{ and } \||\tau| - \bar{c}| \leqslant \beta\}.$$

where

$$\frac{\bar{a}}{2} + (\bar{c} + \beta) < \bar{b}, \quad \text{and} \quad (\bar{b} + \bar{a}) - (\bar{c} - \beta) < \frac{\bar{c} - \beta}{10}. \tag{3.16}$$

(As for the definition of $\mathbb{A}_r$ in the previous subsection, the divisor "10" is significant in only that $10 > 2$.)

The conditions on $\bar{\mathbb{A}}$ ensure in explicit terms that all lengths of $\tau \subset I$ are possible and that $|\tau'| \leqslant |\tau|/10$. Indeed, $-\bar{a} < -\bar{a}/2 \leqslant l$ and, by the first inequality in (3.16), $l + |\tau| < \bar{a}/2 + (\bar{c} + \beta) < \bar{b}$ for all lengths. Also, since $|\tau| \geqslant \bar{c} - \beta$ we have, by the second inequality in (3.16), that $|\tau'| \leqslant (\bar{b} + \bar{a}) - (\bar{c} - \beta) < (\bar{c} - \beta)/10 \leqslant |\tau|/10$. In addition, parameters $\bar{a}, 2\beta \leqslant \bar{c}/10$ and $\bar{b} < \bar{c} + \bar{c}/10$: Combining the inequalities of (3.16), $3\bar{a}/2 + 2\beta < \bar{b} + \bar{a} - (\bar{c} - \beta) < \bar{c}/10$ which gives the bounds. Then, also $|l| \leqslant \tilde{c}/2$ and $\bar{c} - \tilde{c}/2 < |\tau| < \bar{c} + \tilde{c}/2$ where $\tilde{c} = \bar{c}/10$ for convenience.

The parameters give the picture

$$\begin{aligned} -\tilde{c} &< -\bar{a} < -\bar{a}/2 \leqslant l \leqslant \bar{a}/2 < \bar{a} < \tilde{c} \\ &< -\tilde{c}/2 + (\bar{c} - \beta) < l + |\tau| < \bar{a}/2 + (\bar{c} + \beta) < \bar{b} < \bar{c} + \tilde{c}. \end{aligned} \tag{3.17}$$

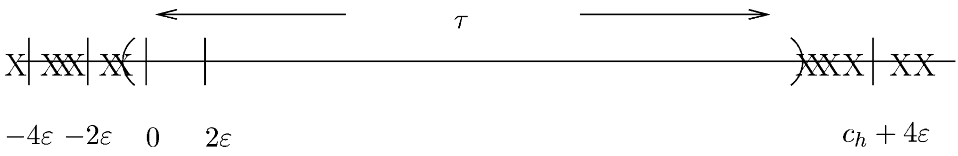

Fig. 2. A typical scaled picture.

In the following, the parameter $\bar{c}$ will always be fixed although $\bar{a}, \bar{b}$ and β will be allowed to vary within the confines of (3.16). One should think of $\bar{c}$ as the clearing interval "length" comparable to the interval length $\bar{b}+\bar{a}$, and $\bar{a}, \beta$ and $\bar{b}-\bar{c}$ as much smaller numbers. The typical choice to keep in mind (which is made use of in the proof of Theorem 2.2) is when $\bar{c}=c_h$, and $\bar{a}, \bar{b}, \alpha$ and β are all given in terms of a small number $\varepsilon > 0$, namely $\bar{a} = 4\varepsilon$, $\bar{b} = c_h + 4\varepsilon$, and $\beta = \varepsilon$ (Fig. 2).

Proposition 3.2. *Let $f_u : C_b(\Omega_u) \to \mathbb{R}$ be a bounded continuous function on continuous paths up to time u. Then, we have*

$$\overline{\lim_{\bar{a},\beta\downarrow 0}}\ \overline{\lim_{s\uparrow\infty}} \sup_{x\in(-\bar{a},\bar{a})} \sup_{\omega\in\bar{\mathbb{A}}} |\bar{E}^I_{\omega,x,s}[f_u(X(\cdot))] - E_0^{(0,\bar{c})}[f_u(X(\cdot))]| = 0.$$

We defer the proof of the proposition until after a series of lemmas.

The scheme of the proof is first to approximate $\bar{E}^I_{\omega,x,s}[f_u]$ by $\bar{E}^I_{\omega,q,s}[f_u]$ for a $q \in (0,\tilde{c}]$ and small $\bar{a}$ (Lemmas 3.9–3.11). Then, $\bar{E}^I_{\omega,q,s}[f_u]$ is approximated by $E^\tau_q[f_u]$ (Lemma 3.7). Next, $E^\tau_q[f_u]$ is approximated by $E_q^{(0,\bar{c})}[f_u]$ (Lemma 3.8). Finally, Proposition 3.1 will give that $E_q^{(0,\bar{c})}[f_u]$ is close to $E_0^{(0,\bar{c})}[f_u]$ for small q.

Also, with respect to the application of results in the previous subsection for the proof of Proposition 3.2, it will be helpful to note that V_s acts on $s\omega$ in the definition of $\bar{V}_s$ and $(s\omega)_s = \omega$.

Lemma 3.7. *Let f_u be a bounded $\mathscr{F}_u$-measurable function. Then, for $q \in [\bar{a}, \tilde{c}]$ we have*

$$\overline{\lim_{s\to\infty}} \sup_{\omega\in\bar{\mathbb{A}}} |\bar{E}^I_{\omega,q,s}[f_u] - E^\tau_q[f_u]| = 0.$$

Proof. Since $l < \bar{a} < \tilde{c} < l + |\tau|$, we have $\tau \supset [\bar{a}, \tilde{c}]$ (cf. (3.17)). Therefore, $\phi_1^\tau(q) > 0$ for $\bar{a} \leqslant q \leqslant \tilde{c}$, and we can write the expression in absolute value above as

$$\exp\{\lambda_1^{I,\bar{V}_s}u\}E_q\left[f_u, \exp\left\{-\int_0^u \bar{V}_s\,\mathrm{d}r\right\}, T_I > u, \phi_1^{I,\bar{V}_s}(X(u))\right] / \phi_1^{I,\bar{V}_s}(q)$$
$$-\exp\{\lambda_1^\tau u\}E_q[f_u, T_\tau > u, \phi_1^\tau(X(u))]/\phi_1^\tau(q).$$

For $\omega \in \bar{\mathbb{A}}$, we note $\phi_1^\tau = \phi_1^{I,\omega}$ and $\lambda_1^\tau = \lambda_1^{I,\omega}$. Then, as remarked earlier, since V_s acts on $s\omega$ in the definition of $\bar{V}_s$ and $(s\omega)_s = \omega$, by Lemma 3.6, the denominators are bounded away from zero and converge uniformly over $\omega \in \bar{\mathbb{A}}$. The exponential factors, similarly, in the numerators also converge uniformly over $\bar{\mathbb{A}}$ using Lemma 3.3.

It remains to show that the expectations converge uniformly also. Let $c = |\tau|$ and write, for $\varepsilon > 0$, that

$$\begin{aligned}
&|E_q[f_u, \mathrm{e}^{-\int_0^u \bar{V}_s\,\mathrm{d}r}, T_I > u, \phi_1^{I,\bar{V}_s}(X(u))] - E_q[f_u, T_\tau > u, \phi_1^\tau(X(u))]| \\
&\quad\leqslant |E_q[f_u, \mathrm{e}^{-\int_0^u \bar{V}_s\,\mathrm{d}r}, T_I > u, \phi_1^{I,\bar{V}_s}(X(u))] - E_q[f_u, T_{(\varepsilon,c-\varepsilon)+l} > u, \\
&\phi_1^{(\varepsilon,c-\varepsilon)+l}(X(u))]| + |E_q[f_u, T_{(\varepsilon,c-\varepsilon)+l} > u, \phi_1^{(\varepsilon,c-\varepsilon)+l}(X(u))] \\
&\quad -E_q[f_u, T_\tau > u, \phi_1^\tau(X(u))]|. \qquad (3.18)
\end{aligned}$$

We now estimate the two resulting absolute values. We borrow the technique from p. 1165 of Sznitman (1991). For the first term, note that $\bar{V}_s = 0$ on $(l + C(W)/s, l + c - C(W)/s) \supset (\varepsilon, c-\varepsilon) + l$ for large s and therefore $\exp\{-\int_0^u \bar{V}_s(X(r),\omega)\,\mathrm{d}r\} I(T_I > u) \geqslant I(T_{(\varepsilon,c-\varepsilon)+l} > u)$. We bound the first term above then by

$$\begin{aligned}
&\|f\|_{L^\infty}(E_q[(\mathrm{e}^{-\int_0^u \bar{V}_s\,\mathrm{d}r} I(T_I > u) - I(T_{(\varepsilon,c-\varepsilon)+l} > u))\phi_1^{I,\bar{V}_s}(X(u))] \\
&\quad + E_q[I(T_{(\varepsilon,c-\varepsilon)+l} > u)|\phi_1^{I,\bar{V}_s} - \phi_1^{(\varepsilon,c-\varepsilon)+l}|(X(u))]) \\
&\qquad \leqslant \|f\|_{L^\infty}(2\|\phi_1^{I,\bar{V}_s} - \phi_1^{(\varepsilon,c-\varepsilon)+l}\|_{L^\infty} + |E_q[\mathrm{e}^{-\int_0^u \bar{V}_s\,\mathrm{d}r} I(T_I > u)\phi_1^{I,\bar{V}_s}(X(u))] \\
&\qquad\quad - E_q[I(T_{(\varepsilon,c-\varepsilon)+l} > u)\phi_1^{(\varepsilon,c-\varepsilon)+l}(X(u))]|) \\
&\qquad = \|f\|_{L^\infty}(2\|\phi_1^{I,\bar{V}_s} - \phi_1^{(\varepsilon,c-\varepsilon)+l}\|_{L^\infty} \\
&\qquad\quad + |\phi_1^{I,\bar{V}_s}(q)\mathrm{e}^{-\lambda_1^{I,\bar{V}_s}u} - \phi_1^{(\varepsilon,c-\varepsilon)+l}(q)\mathrm{e}^{-\lambda_1^{(\varepsilon,c-\varepsilon)+l}u}|).
\end{aligned}$$

Observe now, from (2.1), that uniformly over $s \geqslant 1$ and c such that $|c - \bar{c}| \leqslant \beta$ we have $\|\phi_1^\tau - \phi_1^{(\varepsilon,c-\varepsilon)+l}\|_{L^\infty} = \mathrm{O}(\varepsilon)$ and $|\lambda_1^\tau - \lambda_1^{(\varepsilon,c-\varepsilon)+l}| = \mathrm{O}(\varepsilon)$ as $\varepsilon \downarrow 0$. Therefore, by Lemmas 3.6 and 3.3, the last quantity above, as $s \uparrow \infty$, is $\mathrm{O}(\varepsilon)$ for small ε.

The last term in (3.18) is bounded similarly $\mathrm{O}(\varepsilon)$ as $\varepsilon \downarrow 0$. This finishes the proof as ε is arbitrary. □

An estimate in a similar vein is the following.

Lemma 3.8. *Let f_u be a bounded $\mathscr{F}_u$-measurable function, and let $q \in (0, \tilde{c}]$. Then,*

$$\sup_{s\geqslant 1} \sup_{\omega\in\bar{\mathbb{A}}} |E_q^\tau[f_u] - E_q^{(0,\bar{c})}[f_u]| = \mathrm{O}(\bar{a}) + \mathrm{O}(\beta) \quad \textit{as } \bar{a} \to 0 \textit{ and } \beta \to 0.$$

Proof. Let $s \geqslant 1$ and $\omega \in \bar{\mathbb{A}}$. When $\bar{a}/2 < q$, we have $l < q \leqslant \tilde{c}$ so that $q \in \tau$ (cf. (3.17)). Therefore $\phi_1^\tau(q) > 0$ and $E_q^\tau[f_u]$ makes sense for all small $\bar{a}$. Let now $c = |\tau|$ and $G = (-\bar{a}, \bar{c} + \beta + \bar{a}/2)$. Then, the bound $\sin(\pi x/d)1_{[0,d]}(x) \leqslant (d'/d)\sin(\pi x/d')1_{[0,d']}(x)$, for $d' \geqslant d > 0$, the observations $\tau \subset G$ and $|\tau| \geqslant \bar{c} - \beta$ (cf. (3.17)), and (2.1) lead to the following:

$$I(T_\tau > u), I(T_{(0,\bar{c})} > u) \leqslant I(T_G > u) \quad \text{and} \quad \phi_1^\tau, \phi_1^{(0,\bar{c})} \leqslant (|G|/(\bar{c} - \beta))^{3/2}\phi_1^G.$$

Now bound the difference $|\bar{E}_q^\tau[f_u] - \bar{E}_q^{(0,\bar{c})}[f_u]|$, with the technique of Lemma 3.7, by

$$\begin{aligned}
&\frac{e^{\lambda^\tau u}}{\phi_1^\tau(q)}|E_q[f_u, T_\tau > u, \phi_1^\tau(X(u))] - E_q[f_u, T_G > u, (|G|/(\bar{c}-\beta))^{3/2}\phi_1^G(X(u))]| \\
&\quad + \left| \frac{e^{\lambda^\tau u}}{\phi_1^\tau(q)} E_q[f_u, T_G > u, (|G|/(\bar{c}-\beta))^{3/2}\phi_1^G(X(u))] \right. \\
&\quad \left. - \frac{e^{\lambda^{(0,\bar{c})} u}}{\phi_1^{(0,\bar{c})}(q)} E_q[f_u, T_{(0,\bar{c})} > u, \phi_1^{(0,\bar{c})}(X(u))] \right| \\
&\quad \leqslant \|f\|_{L^\infty} \left[\frac{e^{\lambda^\tau u}}{\phi_1^\tau(q)}((|G|/(\bar{c}-\beta))^{3/2}\phi_1^G(q)e^{-\lambda_1^G u} - \phi_1^\tau(q)e^{-\lambda^\tau u}) \right. \\
&\quad + (|G|/(\bar{c}-\beta))^{3/2}\phi_1^G(q)e^{-\lambda^G u}\left| \frac{e^{\lambda^\tau u}}{\phi_1^\tau(q)} - \frac{e^{\lambda^{(0,\bar{c})} u}}{\phi_1^{(0,\bar{c})}(q)} \right| \\
&\quad \left. + \frac{e^{\lambda^{(0,\bar{c})} u}}{\phi_1^{(0,\bar{c})}(q)}((|G|/(\bar{c}-\beta))^{3/2}\phi_1^G(q)e^{-\lambda^G u} - \phi_1^{(0,\bar{c})}(q)e^{-\lambda^{(0,\bar{c})} u}) \right].
\end{aligned}$$

Using (2.1), the above expression is $O(\bar{a}) + O(\beta)$ uniformly over $s \geqslant 1$ and $\omega \in \bar{\mathbb{A}}$ as $\bar{a}$ and β vanish. This finishes the proof. □

Define now, for numbers $\varepsilon, \delta, \mu > 0$, the set $\hat{K}(\varepsilon, \delta, \mu) \subset \Omega$ by its complement,

$$\hat{K}^c(\varepsilon, \delta, \mu) = \left\{ \sup_{\substack{0 \leqslant r,t \leqslant \mu \\ |r-t| < \delta}} |X(r) - X(t)| > \varepsilon \right\}.$$

Then, sets of the form $K(\varepsilon, \delta, \mu, \rho) = \{X(0) \leqslant \rho\} \cap \hat{K}(\varepsilon, \delta, \mu)$ are compact in Ω for $\rho > 0$.

Lemma 3.9. *For $\mu > 0$, we have that*

$$\overline{\lim_{\delta \to 0}}\, \overline{\lim_{\bar{a} \to 0}}\, \overline{\lim_{s \to \infty}} \sup_{x \in (-\bar{a}, \tilde{c}]} \sup_{\omega \in \bar{\mathbb{A}}} \bar{P}^I_{\omega,x,s}(K^c(3\tilde{c}, \delta, \mu, \tilde{c})) = 0.$$

Proof. We remark that the limit on $\bar{a}$ does not play a role in the following argument. But we include it in this order for the proof later of Proposition 3.2.

First, for $x \in (-\bar{a}, \tilde{c}]$, we have that $\bar{P}^I_{\omega,x,s}(K^c(3\tilde{c}, \delta, \mu, \tilde{c})) = \bar{P}^I_{\omega,x,s}(\hat{K}^c(3\tilde{c}, \delta, \mu))$. Then, also as $x \in (-\bar{a}, \tilde{c}]$ we have from the strong Markov property that

$$\bar{P}^I_{\omega,x,s}(\hat{K}^c(3\tilde{c}, \delta, \mu)) \leqslant \bar{P}^I_{\omega,x,s}\left(\sup_{\substack{0\leqslant r,t\leqslant\mu\\ |r-t|<\delta}} |X(r+\mathscr{H}_{\tilde{c}}) - X(t+\mathscr{H}_{\tilde{c}})| > \tilde{c}\right)$$

$$= P_{\omega,\tilde{c},s}(\hat{K}^c(\tilde{c}, \delta, \mu)).$$

Let $K_1 = \hat{K}(\tilde{c}, \delta, \mu)$ and observe, from Lemma 3.7 (with $u = \mu$ and $f_u = I(K_1^c)$), that

$$\overline{\lim_{s\to\infty}} \sup_{\omega\in\bar{\mathbb{A}}} |\bar{E}^I_{\omega,\tilde{c},s}[I(K_1^c)] - E^{\tau}_{\tilde{c}}[I(K_1^c)]| = 0.$$

Crude bounds now suffice. Let $G = (-\tilde{c}, \bar{c} + \tilde{c})$ and $J = (\tilde{c}/2, \bar{c} + \tilde{c})$. Then, for $\omega \in \bar{\mathbb{A}}$, we have that

$$E^{\tau}_{\tilde{c}}[I(K_1^c)] = \frac{e^{\lambda_1^{\tau}\mu}}{\phi_1^{\tau}(\tilde{c})} E_{\tilde{c}}[I(K_1^c) I(T_{\tau>\mu}) \phi_1^{\tau}(X(\mu))]$$

$$\leqslant C \frac{\phi_1^G(\tilde{c})}{\phi_1^J(\tilde{c})} \frac{e^{\pi^2\mu/(2(\bar{c}-\tilde{c})^2)}}{e^{\pi^2\mu/(2(2\tilde{c}+\bar{c})^2)}} E^G_{\tilde{c}}[I(K_1^c)].$$

For the last inequality, we have used that $|l| < \tilde{c}/2$ and $\bar{c} - \tilde{c}/2 \leqslant |\tau| \leqslant \bar{c} + \tilde{c}/2$ from the comments near (3.17) to deduce from (2.1) that $\phi_1^{\tau}(\tilde{c}) \geqslant \phi_1^J(\tilde{c})$. Also, we use $e^{\lambda_1^{\tau}\mu} \leqslant e^{\pi^2\mu/(2(\bar{c}-\tilde{c}/2)^2)}$ and, as $\tau \subset G$ from (3.17), that $I(T_\tau > t) \leqslant I(T_G > t)$ and $\phi_1^{\tau} \leqslant C\phi_1^G$ for a $C = C(\bar{c})$ determined from (2.1).

The lemma now follows as $P^G_{\tilde{c}}$ is concentrated on continuous paths. □

Lemma 3.10. *There exists $s_1 = s_1(W, \bar{c}) \geqslant 1$ such that for $s \geqslant s_1$ and all $\omega \in \bar{\mathbb{A}}$, we have that*

$$\phi_1^{I,\bar{V}_s}(x) \leqslant \phi_1^{I,\bar{V}_s}(y) \quad \textit{when} \quad -\bar{a} < x \leqslant y \leqslant \tilde{c}.$$

Proof. To simplify the exposition, let us call $\phi = \phi_1^{I,\bar{V}_s}$ and $\lambda = \lambda_1^{I,\bar{V}_s}$. As ϕ is positive on I, we may form $u = (\mathrm{d}/\mathrm{d}x)(\log\phi(x))$ and compute that it satisfies on I,

$$u' = (\bar{V}_s - \lambda) - u^2.$$

Then, we have $\phi(x) = C\exp\{\int^x u(y)\,\mathrm{d}y\}$ and $\phi'(x) = u(x)\phi(x)$.

To prove the lemma, it suffices to show that u is of one sign on $(-\bar{a}, \tilde{c}]$: If so, as $\phi(-\bar{a}) = 0$ and $\phi(\tilde{c}) > 0$, we conclude that $u(x) \geqslant 0$ and so $\phi'(x) \geqslant 0$ for $x \in (-\bar{a}, \tilde{c}]$ implying that ϕ increases on this set.

To show that u is either positive or negative on $(-\bar{a}, \tilde{c}]$, we prove that u cannot vanish on this domain. Suppose otherwise, and let $x_0 \in (-\bar{a}, \tilde{c}]$ be a point where $u(x_0) = 0$. Consider the initial value problem for $v(x) = u(x_0 - x)$: $v(0) = 0$ and

$$v' = -(\bar{V}_s(x_0 - x) - \lambda) + v^2$$

$$\leqslant \lambda + v^2.$$

Explicitly solving, we have that $v(x) \leqslant \sqrt{\lambda}\tan(\sqrt{\lambda}x)$ for $x \in [0, \pi/(2\sqrt{\lambda}))$. Therefore,

$$\begin{aligned}
\phi(x_0 - x) &= \phi(x_0)\exp\left\{\int_{x_0}^{x_0-x} u(y)\,\mathrm{d}y\right\} \\
&= \phi(x_0)\exp\left\{-\int_0^x u(x_0 - y)\,\mathrm{d}y\right\} \\
&\geqslant \phi(x_0)\exp\{\log[\cos(\sqrt{\lambda}y)]|_0^x\} \\
&= \phi(x_0)\cos(\sqrt{\lambda}x).
\end{aligned}$$

Now, noting that V_s acts on $s\omega$ in the definition of $\bar{V}_s$ and $(s\omega)_s = \omega$, calculate from Lemma 3.3 for s large and the fact $|\tau| > \bar{c} - \tilde{c}/2$ (cf. (3.17)) that $\pi/(2\sqrt{\lambda}) > (\bar{c} - \tilde{c}/2)/2$. Therefore, as $\cos(\sqrt{\lambda}x) > 0$ for $x \in [0, (\bar{c} - \tilde{c}/2)/2) \subset [0, \pi/(2\sqrt{\lambda}))$, we have that $\phi(x_0 - x)$ also stays positive on this set. We force a contradiction however as $\phi(x_0 - (x_0 + \bar{a})) = 0$ and, $0 < x_0 + \bar{a} < 2\tilde{c} < (\bar{c} - \tilde{c}/2)/2$ from the inequality $\bar{a} < \tilde{c}$ and definition $\tilde{c} = \bar{c}/10$ (cf. near (3.17)). This finishes the proof. □

Lemma 3.11. *For all $s \geqslant s_1(W, \bar{c})$ as in Lemma 3.10, and all $q \in [\bar{a}, \tilde{c}]$, we have that*

$$\sup_{x\in(-\bar{a},\bar{a})}\ \sup_{\omega\in\bar{\mathbb{A}}} \bar{E}^I_{\omega,x,s}[\mathscr{H}_q] \leqslant 2(q + \bar{a})^2.$$

Proof. We follow the proof of Lemma 4 (Povel, 1995). The method in Povel (1995) is to represent $\bar{E}^I_{\omega,x,s}[\mathscr{H}_q]$ in terms of the speed and scale measures and associated Green's function of the process. (cf. Karatzas and Shreve, 1991, 5.5 B,C for definitions). Let $m(dy)$ be the speed measure, and G the Green's function. We have, for $-\bar{a} < a^* < x < \bar{a}$, that

$$\begin{aligned}
\bar{E}^I_{\omega,x,s}[\mathscr{H}_q] &= \lim_{a^*\downarrow -\bar{a}} \bar{E}^I_{\omega,x,s}[T_{(a^*,q)}]] \quad (\text{as } \bar{P}^I_{\omega,x,s}(\mathscr{H}_{-\bar{a}} < \mathscr{H}_q) = 0) \\
&= \lim_{a^*\downarrow -\bar{a}} \int_{a^*}^q G_{(a^*,q)}(x,y)m(d\,y).
\end{aligned}$$

To evaluate these quantities explicitly, recall the drift $\mathrm{d}(z) = \mathrm{d}/\mathrm{d}z \log \phi_1^{I,\bar{V}_s}(z)$ of the taboo process (3.15). For simplicity, let $\phi = \phi_1^{I,\bar{V}_s}$ and let $\gamma \in [\tilde{c}, \bar{c} - \beta - \tilde{c}/2] \subset \tau$ (cf. (3.17)) so that $\phi(\gamma) > 0$. Then, for $z \in (-\bar{a}, \bar{b})$, the scale function $s(z)$ is defined

$$\begin{aligned}
s(z) &= \int_\gamma^z \exp\left\{-2\int_\gamma^l \mathrm{d}(r)\,\mathrm{d}r\right\}\mathrm{d}l \\
&= \phi(\gamma)^2 \int_\gamma^z \frac{1}{\phi(l)^2}\,\mathrm{d}l.
\end{aligned}$$

Correspondingly, the speed and Green's functions are given as

$$m(\mathrm{d}z) = 2(s'(z))^{-1} = \frac{2\phi(z)^2}{\phi(\gamma)^2}\,\mathrm{d}z,$$

$$G_{(a^*,q)}(x,y) = \frac{(s(\min(x,y)) - s(a^*))(s(q) - s(\max(x,y)))}{s(q) - s(a^*)}.$$

Now note for $a^* < y \leqslant q$ that $a^* < \min(x,y) \leqslant q$ and therefore

$$\frac{s(\min(x,y)) - s(a^*)}{s(q) - s(a^*)} \leqslant 1.$$

Putting these observations together, we have

$$\begin{aligned}\bar{E}^I_{\omega,x,s}[\mathscr{H}_q] &\leqslant \frac{2}{\phi(\gamma)^2}\int_{-\bar{a}}^{q} (s(q) - s(\max(x,y)))\phi(y)^2\,\mathrm{d}y\\ &= 2\int_{-\bar{a}}^{q}\int_{\max(x,y)}^{q} \frac{\phi(y)^2}{\phi(z)^2}\,\mathrm{d}z\,\mathrm{d}y.\end{aligned}$$

From Lemma 3.10, we have for $s \geqslant s_1(W,\bar{c})$ that $\phi(y)^2/\phi(z)^2 \leqslant 1$ for $-\bar{a} < y \leqslant z \leqslant \tilde{c}$. This gives $\bar{E}^I_{\omega,x,s}[\mathscr{H}_q] \leqslant 2(q+\bar{a})^2$ to finish the lemma. □

Proof of Proposition 3.2. Let $q \in (0,\tilde{c}]$, and consider $\bar{a}$ such that $\bar{a} < 2q$. Let $x \in (-\bar{a},\bar{a})$, and $\omega \in \bar{\mathbb{A}}$ for large $s \geqslant 1$. Write

$$\begin{aligned}|\bar{E}^I_{\omega,x,s}[f_u] - E_0^{(0,\tilde{c})}[f_u]| &\leqslant |\bar{E}^I_{\omega,x,s}[f_u] - \bar{E}^I_{\omega,q,s}[f_u]| + |\bar{E}^I_{\omega,q,s}[f_u] - E_q^{\tau}[f_u]|\\ &\quad + |E_q^{\tau}[f_u] - E_q^{(0,\tilde{c})}[f_u]| + |E_q^{(0,\tilde{c})}[f_u] - E_0^{(0,\tilde{c})}[f_u]|\\ &= J_1 + J_2 + J_3 + J_4.\end{aligned}$$

We now handle each term separately.

J_1: For $\mu > 0$, let $K = K(3\tilde{c},\delta,\mu,\tilde{c}) \subset \Omega$ be the compact set in Lemma 3.9. The set K satisfies $\bar{P}^I_{\omega,x,s}(K^c) < \varepsilon(\delta,\bar{a},s;\tilde{c},\mu)$ for all $|x| < \bar{a}(<\tilde{c})$ where $\lim \varepsilon(\delta,\bar{a},s;\tilde{c},\mu) = 0$ and $\lim = \lim_{\delta\downarrow 0}\lim_{\bar{a}\downarrow 0}\lim_{s\uparrow\infty}$.

For a path $w \in \Omega$, let $\theta_t(w) = w(\cdot + t)$ denote the t-shift on Ω. Let also $0 < \gamma < 1$, and $0 \leqslant r \leqslant \gamma$. Note, as f_u is continuous on Ω, that f_u is uniformly continuous on K and $|f_u - f_u \circ \theta_r|I(K) \leqslant \varepsilon(\gamma)$, where $\varepsilon(\gamma) = \varepsilon(\gamma;\tilde{c},\delta,\mu,f_u) \downarrow 0$ as $\gamma \downarrow 0$.

Then, from the Markov property and Lemma 3.11, we have that

$$\begin{aligned}J_1 &= |\bar{E}^I_{\omega,x,s}[f_u] - \bar{E}^I_{\omega,x,s}[f_u \circ \theta_{\mathscr{H}_q}]|\\ &\leqslant \bar{E}^I_{\omega,x,s}[|f_u - f_u\circ\theta_{\mathscr{H}_q}|I(\mathscr{H}_q > \gamma)] + \bar{E}^I_{\omega,x,s}[|f_u - f_u\circ\theta_{\mathscr{H}_q}|I(\mathscr{H}_q \leqslant \gamma)]\\ &\leqslant 2\|f_u\|_{L^\infty}\frac{2(q+\bar{a})^2}{\gamma} + (2\|f_u\|_{L^\infty})\varepsilon(\delta,\bar{a},s;\tilde{c},\mu)\\ &\quad + \bar{E}^I_{\omega,x,s}[|f_u - f_u\circ\theta_{\mathscr{H}_q}|I(\mathscr{H}_q \leqslant \gamma)I(K)]\\ &\leqslant 2\|f_u\|_{L^\infty}\frac{2(q+\bar{a})^2}{\gamma} + (2\|f_u\|_{L^\infty})\varepsilon(\delta,\bar{a},s;\tilde{c},\mu) + \varepsilon(\gamma;\bar{c},\delta,\mu,f_u).\end{aligned}$$

J_2: By Lemma 3.7, $J_2 \leqslant \varepsilon(s;q,f_u)$ where $\varepsilon(s;q,f_u) \to 0$ as $s \to \infty$.

J_3: By Lemma 3.8, $J_3 = \mathrm{O}(\bar{a}) + \mathrm{O}(\beta)$ uniformly for $s \geqslant 1$ as $\bar{a}$ and β vanish.

J_4: By Proposition 3.1, $J_4 = \varepsilon(q;f_u)$ where $\varepsilon(q;f_u) \to 0$ as $q \to 0$.

To finish the proof of the proposition, take limit first on $s \uparrow \infty$, then on $\bar{a}, \beta \downarrow 0$, then on $q \downarrow 0$, then on $\gamma \downarrow 0$, and finally on $\delta \downarrow 0$. $\square$

4. Forest-clearing coarse grained picture

To deduce Proposition 2.1, we will need some refinements of the estimates Povel uses to prove the large deviation upper bounds mentioned in the introduction near (1.4). Namely, the bound for $y \in \mathbb{R}$,

$$\overline{\lim_{t\to\infty}}\, t^{-1/3} \log \mathbb{E} \otimes E_0 \left[X(t) \in B(yt^{1/3}), \exp\left\{ -\int_0^t V(X(s),\omega)\,\mathrm{d}s \right\} \right] \leqslant I(y), \tag{4.1}$$

where $B(x)$ is the 1-neighborhood of x. We now describe briefly the set-up of Section 2, Povel (1997). We will refer the reader to Povel (1997) for fuller explanations of some statements.

The first step in the proof of the upper bounds is to restrict the motion to the interval $I_M = (-Mt^{1/3}, Mt^{1/3})$, for $M > \max(I(y)/\beta_0(1), |y|)$ large (cf. Povel, 1997, Lemma 2.1 which makes use of (1.3)).

The strategy now is to describe regions in I_M where Poisson points are "sparse" and the complements of these regions where the points are "dense." To this end, chop $\mathbb{R}$ into intervals of length t^δ where $\delta \in (1/6, 1/3)$. Further chop these intervals into subboxes of length $3a$ where a is the radius of support of W. Pick now a parameter $\alpha \in (0, \frac{1}{3}a)$. With respect to a configuration ω, if the number of subboxes in an interval receiving a Poisson point is less than αt^δ, then declare the interval to be a "thin edge." If instead the number of subboxes is larger than αt^δ, then call the interval simply as an "edge." This construction partitions $\mathbb{R}$ into thin edges and edges.

Now select a parameter $r \in (0, M)$, and consider those connected components of thin edges, $\{L^i\}$, such that $|L^i| \geqslant rt^{1/3}$ (note at the ends of each L^i there are edges). At this point, look at the open t^δ neighborhood Θ of $\bigcup_i L^i$ and call the connected components of Θ as "pseudo-holes," $\{\Theta_i\}$. For fixed r, the number of pseudo-holes intersecting I_M is less than $2M/r + 1$.

With this coarse-grained picture, intuition now dictates that the surviving Brownian traveler spends most of the time in pseudo-holes and does not leave the pseudo-hole set often. This can be formalized to an extent. Let L_t be the fraction of time the process, up to time t, spends outside the pseudo-hole set Θ. Let also $N(t)$ be the number of times the process, up to time t, enters $\bigcup_i L^i$ and exits Θ (so that on each trip it passes over an edge which is costly). Typically, $L_t < \eta$, for some small $\eta \in (0,1)$, and $N(t) \leqslant [t^\delta]$. See Section 2 and Propositions 2.1 and 2.2 (Povel, 1997) for precise definitions and statements.

To gain further insight into the typical path structure, let l denote the total space in the visited pseudo-holes up to time t. More carefully, for some $1 \leqslant K \leqslant 2M/r + 1$, let $\{\Theta_i\colon\ i = 1, \ldots K\}$ be those pseudo-holes, intersecting I_M, which the process visits up to time t. Then $l = \sum_{i=1}^{K} |\Theta_i \cap I_M|$.

Also, we define l' to be the length of the largest interval empty of Poisson points in the visited pseudo-hole set: $l' = |\tau|(\bigcup_{i=1}^{K} \Theta_i \cap I_M, \omega)$. Clearly,

$$l' \leqslant \sum_{i=1}^{K} |\Theta_i \cap I_M| = l. \tag{4.2}$$

Let now $x_i - t^\delta, y_i + t^\delta \in t^\delta \mathbb{Z}$ be the points which mark the visited pseudo-hole set in I_M and satisfy

$$x_1 < y_1 < x_2 < \cdots < x_K < y_K$$

and

$$x_K < Mt^{1/3}, \quad y_1 > -Mt^{1/3}, \quad y_i - x_i \geqslant rt^{1/3}, \quad x_{i+1} - y_i \geqslant 2t^\delta. \tag{4.3}$$

Relabel the visited pseudo-holes so that $\Theta_i = (x_i - t^\delta, y_i + t^\delta)$ for $i = 1, \ldots, K$. It will be helpful to consider the case when we know, in addition to $X(0) = 0$ and $X(t) \in B(yt^{1/3})$, that also the 1-neighborhood of $zt^{1/3}$ is hit before time t, $\mathscr{H}(zt^{1/3}) < t$ for some $zt^{1/3} \in I_M$; of course, when $z = 0$ this is no restriction.

Let now $|u_i - v_i|$ represent the length of the "forest" between the ith and $(i+1)$th pseudo-holes. More precisely, if $y > 0$, let

$$u_i = y_i + t^\delta \quad \text{for } 1 \leqslant i \leqslant K \quad \text{and} \quad v_i = x_{i+1} \quad \text{for } 0 \leqslant i \leqslant K-1,$$

$$u_0 = \begin{cases} zt^{1/3} & \text{when } zt^{1/3} < \min\{0, x_1 - 2t^\delta\} \\ \min\{0, v_0\} & \text{otherwise,} \end{cases}$$

$$v_K = \begin{cases} \max\{y,z\}t^{1/3} & \text{when } \max\{y,z\}t^{1/3} > y_K + 2t^\delta \\ u_K & \text{otherwise.} \end{cases}$$

Analogously, if $y \leqslant 0$, define $u_i = x_{K-i+1} - t^\delta$ for $1 \leqslant i \leqslant K$ and $v_i = y_{K-i}$ for $0 \leqslant i \leqslant K-1$, and also $u_0 = zt^{1/3}$ when $zt^{1/3} > \max\{0, y_K + 2t^\delta\}$ and $u_0 = \max\{0, v_0\}$ otherwise, and $v_K = \min\{y,z\}t^{1/3}$ when $\min\{y,z\}t^{1/3} < x_1 - 2t^\delta$ and $v_K = u_K$ otherwise. Define now $\tilde{l}$ to be the length of forest space traveled, $\tilde{l} = \sum_{i=0}^{K} |u_i - v_i|$ (Fig. 3). Note, with some easy calculation (see Povel, 1997, near (2.80)), that

$$\tilde{l} \geqslant \left[|y|t^{1/3} - \sum_{i=1}^{K} |\Theta_i \cap I_M| \right]_+ - t^\delta = [|y|t^{1/3} - l]_+ - t^\delta. \tag{4.4}$$

Finally, we mention that, due to the coarse-graining, there are at most

$$\exp\{\mathscr{C}_1(t, \delta, M, r)\}, \quad \mathscr{C}_1 = \mathrm{o}(t^{1/3}) \tag{4.5}$$

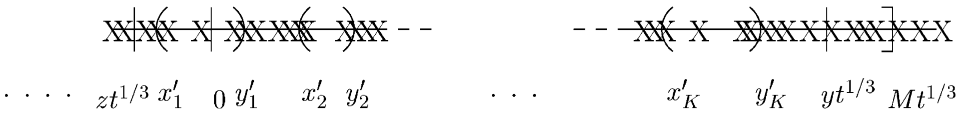

Fig. 3. Possible visited pseudo-hole configuration with $x'_i = x_i - t^\delta$ and $y'_i = y_i + t^\delta$.

possible pseudo-hole configurations as $t \uparrow \infty$ such that $L_t < \eta$, $N(t) \leqslant [t^\delta]$, $T_{I_M} > t$, $X(t) \in B(yt^{1/3})$, and $\mathscr{H}(zt^{1/3}) < t$. For $z = 0$, this result is Lemma 2.2 (Povel, 1997) and for $z \neq 0$ it is not difficult to modify the arguments there. An outline of the proof of this modification is at the end of the section.

Let $A = A(l, l', \tilde{l}, M, y, z, \delta, \alpha, a, \eta, r, t)$ denote one of these configurations so that $\Theta_1, \ldots, \Theta_K$ are the pseudo-holes which are visited and l, l', and $\tilde{l}$ are defined as above. The following proposition follows from the proof of Proposition 2.3 (Povel, 1997). An outline of how the proof follows from statements (2.70), (2.78), (2.86), and (2.87) in (Povel, 1997) is given at the end of the section.

It will be convenient now to define l_t, $\tilde{l}_t$, and l'_t by $l/t^{1/3}$, $\tilde{l}/t^{1/3}$, and $l'/t^{1/3}$, respectively.

Proposition 4.1. *Let $y \in \mathbb{R}$ and $zt^{1/3} \in I_M$, and consider one of the covering sets A defined near (4.5) with parameters M, δ, α, η, and r defined previously. Let also $\gamma > 0$. Then, for all small $\rho_0, \varepsilon_0 > 0$, there exist positive quantities*

$$\mathscr{C}_2(t) = \mathscr{C}_2(t; M, r, \delta, v, a, \rho_0)$$

and

$$\mathscr{C}_3 = \mathscr{C}_3(\alpha, \eta, \beta_0(1), v, a, \rho_0, \varepsilon_0)$$

where $\mathscr{C}_2(t) = \mathrm{o}(t^{1/3})$ as $t \uparrow \infty$ and $\lim_{\rho_0, \varepsilon_0 \downarrow 0} \lim_{\eta \downarrow 0} \lim_{\alpha \downarrow 0} \mathscr{C}_3 = 0$ such that

$$\mathbb{E} \otimes E_0 \left[A, \exp\left\{ -\int_0^t V(X(r))\, \mathrm{d}r \right\}, l'_t \leqslant \gamma \right]$$
$$\leqslant \exp\{\mathscr{C}_2(t)\} \exp\left\{ -t^{1/3} \left[v l_t + \frac{\pi^2}{2\gamma^2} + \beta_0(1) \tilde{l}_t \right] (1 - \mathscr{C}_3) \right\}. \tag{4.6}$$

We remark that the set $R = \{L_t < \eta, N(t) \leqslant [t^\delta], T_{I_M,} > t\}$ is typical in that $\mathbb{E} \otimes E_0[\exp\{-\int_0^t V(X(s), \omega)\, \mathrm{d}s\}, R^c] = \mathrm{o}(\exp\{-t^{1/3} c(1, v)\})$ as $t \uparrow \infty$ (cf. Section 2 (Povel, 1997)). Then, (4.1) and the upperbounds for the large deviation result near (1.4), follow from the above proposition, and statements (4.4), (4.2), and (4.5) (cf. (2.88) (Povel, 1997)).

We now use Proposition 4.1 to deduce confinement properties of the surviving process for the cases $h = 0$ and $0 < |h| < \beta_0(1)$ and establish Proposition 2.1.

Proof of Proposition 2.1 ($h = 0$, (A)). We work in several stages to restrict, the variables l, l' and $\tilde{l}$, on the set of surviving trajectories up to time t, to certain typical values to preserve the leading order estimate (1.5) for $h=0$,

$$\mathbb{E}\otimes E_0\left[\exp\left\{-\int_0^t V(X(r))\,\mathrm{d}r\right\}\right] = \exp\{-t^{1/3}(c(1,v)+\mathrm{o}(1))\}.$$

As the left quantity is the normalization which makes Q_t^0 a conditional probability measure, it makes sense to define "atypical" or not "typical" sets B as those for which the expectation $\mathbb{E}\otimes E_0[B,\exp\{-\int_0^t V(X(r))\,\mathrm{d}r\}] = \mathrm{o}(\exp\{-t^{1/3}c(1,v)\})$ as $t\uparrow\infty$. Let now $\theta>0$ be a small number in comparison to c_0.

Step 1: Let $K(c,\theta)=[-c-\theta,c+\theta]$. We can restrict the values of $t^{-1/3}X(t)$ under Q_t^0 to the set $K(c_0,\theta)$ by the large deviation principle (1.4). For $X(t)\in B(yt^{1/3})$ when $y\in K(c_0,\theta)$ and $\mathscr{H}(z^{1/3})<t$ when $zt^{1/3}\in I_M$, the coarse-grained picture under Q_t^0 is typical when $T_{(-Mt^{1/3},Mt^{1/3})}>t$, $L_t\leqslant\eta$, and $N(t)\leqslant[t^\delta]$, for M large, η small, and $\delta\in(\frac{1}{6},\frac{1}{3})$.

Step 2: Let A be one of the sets described near (4.5). The total number of the various disjoint 1-balls $B(yt^{1/3})\subset t^{1/3}K(c_0,\theta)$ and $B(zt^{1/3})\subset I_M$, and typical sets A is $\mathrm{e}^{\mathrm{o}(t^{1/3})}$ as $t\uparrow\infty$ (cf. (4.5)). We focus then on a given $y\in K(c_0,\theta)$, $zt^{1/3}\in I_M$, and corresponding set A. Our aim will be to describe in successive reductions what the typical points $zt^{1/3}$ and sets A are.

Step 3: By Proposition 4.1, and bounds (4.2) and (4.4), if $|l_t-c_0|>\theta$, then

$$\begin{aligned}&\mathbb{E}\otimes E_0\left[A,\exp\left\{-\int_0^t V\,\mathrm{d}r\right\},|l_t-c_0|>\theta\right]\\&\quad\leqslant \inf\left[\exp\{\mathscr{C}_2(t)\}\exp\left\{-t^{1/3}\left[vx+\frac{\pi^2}{2x^2}+\beta_0(1)(y-x)_+\right](1-\mathscr{C}_3)\right\}\right]\\&\quad=\mathrm{o}(\exp\{-t^{1/3}c(1,v)\})\end{aligned}$$

as $t\uparrow\infty$ where $\inf=\inf_{\substack{y\in[-c_0-\theta,c_0+\theta]\\|x-c_0|>\theta}}$ and the parameters governing $\mathscr{C}_3$ are small enough to deduce the last step.

Step 4: In fact, for sets A when $|l_t-c_0|\leqslant\theta$, we must have $|l_t'-c_0|\leqslant C_1\theta$, say for $C_1=C_1(v,\beta_0(1))\geqslant 1$ large enough, by the same type of reasoning with Proposition 4.1. The largest empty subinterval $\tau=\tau(\bigcup_{i=1}^K\Theta_i\cap I_M,\omega)$ is then well defined.

Step 5: Define the time,

$$S_{C,B}=\inf\{s\in[0,t]:\ \mathrm{dist}(X(s),B)>C\theta t^{1/3}\},$$

of first exit from the $C\theta t^{1/3}$-neighborhood of B. We argue now, for large enough C, that typically $S_{C,\tau}>t$.

Say that $\tau\subset\Theta_i=(x_i-t^\delta,y_i+t^\delta)$ for some $1\leqslant i\leqslant K$. As $|l_t'-l_t|\leqslant(C_1+1)\theta$, τ virtually exhausts the pseudo-hole Θ_i, and furthermore the remaining pseudo-hole set length is bounded above by $(C_1+1)\theta t^{1/3}$. Then, for large enough $C>3(C_1+1)$ say, the condition $S_{C,\tau}\leqslant t$ implies that $S_{C/2,\Theta_i}\leqslant t$, and also, as Θ_i is visited, that the process travels at least a distance $(C/2-(C_1+1))\theta t^{1/3}$ in non-pseudo-hole regions.

Hence, for $zt^{1/3}$ outside the interval $(x_i - t^\delta - (C/2)\theta t^{1/3}, y_i + t^\delta + (C/2)\theta t^{1/3})$ we have the lower bound $\tilde{l} \geqslant (C/2 - C_1 - 1)\theta t^{1/3}$. Therefore, as typically $|l - c_0 t^{1/3}| \leqslant \theta t^{1/3}$ and $|l' - c_0 t^{1/3}| \leqslant C_1 \theta t^{1/3}$, Proposition 4.1 gives that typically we must have $S_{C_2,\tau} > t$ for a $C_2 = C_2(v, \beta_0(1))$ large.

In particular, as the origin must be within $C_2 \theta t^{1/3}$ of the clearing interval, this locates $\tau \subset t^{1/3}[-c_0 - (C_1 + C_2)\theta, c_0 + (C_1 + C_2)\theta]$. Consequently, the process typically stays inside the $(C_1 + C_2)\theta t^{1/3}$-neighborhood of this interval up to time t, $T_{t^{1/3}K(c_0, 2(C_1+C_2)\theta)} > t$.

Step 6: We now argue that τ is the unique empty subinterval in this interval with large length. In fact, we show that the event of a second distinct large empty subinterval, $B = \{|\tau'(t^{1/3}K(c_0, 2(C_1 + C_2)\theta), \omega)| \geqslant C_3 \theta t^{1/3}\}$ for C_3 large, is not typical. Indeed,

$$
\begin{aligned}
&\mathbb{E} \otimes E_0 \left[A, B, \exp\left\{ -\int_0^t V \, \mathrm{d}r \right\}, \tau \subset t^{1/3} K(c_0, 2(C_1 + C_2)\theta), \right. \\
&\qquad \left. |l_t - c_0| \leqslant \theta, |l'_t - c_0| \leqslant C_1 \theta, T_{t^{1/3}K(c_0, 2(C_1+C_2)\theta)} > t \right] \\
&\quad \leqslant \mathbb{E}[\exists \text{ empty intervals } \tau', \tau \subset t^{1/3} K(c_0, 2(C_1 + C_2)\theta), \\
&\qquad C_3 \theta \leqslant t^{-1/3}|\tau'| \leqslant c_0 + C_1\theta, |t^{-1/3}|\tau| - c_0| \leqslant C_1 \theta, \\
&\qquad \| R_t^{t^{1/3}K(c_0, 2(C_1+C_2)\theta), V} 1 \|_{L^\infty}] \\
&\quad \leqslant \exp\{\mathrm{o}(t^{1/3})\} \mathbb{E}[\exists \text{ empty intervals } \tau', \tau \subset t^{1/3} K(c_0, 2(C_1 + C_2)\theta), \\
&\qquad C_3 \theta \leqslant t^{-1/3}|\tau'| \leqslant c_0 + C_1\theta, |t^{-1/3}|\tau| - c_0| \leqslant C_1 \theta, \\
&\qquad \exp\{-\lambda_1^{t^{1/3}K(c_0, 2(C_1+C_2)\theta), V} t\}] \\
&\quad \leqslant \exp\{\mathrm{o}(t^{1/3})\} \exp\{-t^{1/3}(v(c_0 - C_1\theta + C_3\theta) + (1-\theta)\pi^2/[2(c_0 + C_1\theta)^2])\}.
\end{aligned}
$$

This last expression is $\mathrm{o}(\exp\{-t^{1/3}c(1, v)\})$ as $t \uparrow \infty$ for $\theta > 0$ small and $C_3 > C_1 + \pi^2/(2(c_0 + C_1\theta)^2) + C_1\pi^2/c_0^3$.

Here, in the penultimate line we used Lemma 3.1 and the upperbound in Lemma 3.3 (for $s = 1$). In the last line, to evaluate the $\mathbb{E}$-expectation, we discretized $\mathbb{R}$ into divisions of size $t^{-1/9}$, say, so that the combinatorial complexity for the possible positions of τ' and τ in $t^{1/3}K(c_0, 2(C_1 + C_2)\theta)$ is $\mathrm{O}(\mathrm{e}^{\mathrm{o}(t^{1/3})})$. And, this complexity term was put in the pre-factor. Also, we bounded $\lambda_1^{t^{1/3}K(c_0, 2(C_1+C_2)\theta), V} \geqslant (1-\theta)(\pi^2/(2|\tau|^2)) \geqslant (1-\theta)(\pi^2/(2t^{2/3}(c_0 + C_1\theta)^2))$ from Lemma 3.3 $(s = 1)$ for large t.

This finishes the proof of part (A), with $\varepsilon = (C_1 + C_2)\theta$ small, $t^{1/3}\mathbb{I}^0_{\varepsilon,\omega} = \tau$, and $\zeta = C_3/(C_1 + C_2)$. $\square$

Proof of Proposition 2.1 ($h = 0$, (B)). We begin as in part (A) and follow steps 1–4. Let $J_{\theta,t,b} = \{x \in \tau\colon \ \mathrm{dist}(x, \partial\tau) > b\theta t^{1/3}\}$ where $b\theta < (c_0 - C_1\theta)/2$. If we now impose that $T_{J_{\theta,t,b}} > t$, we have, analogous to step 6 of part (A), that

$$\mathbb{E}\otimes E_0\left[A, \exp\left\{-\int_0^t V\,\mathrm{d}r\right\}, |l_t - c_0| \leqslant \theta, |l_t' - c_0| \leqslant C_1\theta, T_{J_{\theta,t,b}} > t\right]$$

$$\leqslant \mathbb{E}[\exists \text{ empty interval } \tau \subset I_M, |t^{-1/3}|\tau| - c_0| \leqslant C_1\theta, \|R_t^{J_{\theta,t,b},V} 1\|_{L^\infty}]$$

$$\leqslant \exp\{\mathrm{o}(t^{1/3})\}\mathbb{E}[\exists \text{ empty interval } \tau \subset I_M,$$

$$|t^{-1/3}|\tau| - c_0| \leqslant C_1\theta, \exp\{-\lambda_1^{J_{\theta,t,b},V} t\}]$$

$$\leqslant \exp\{\mathrm{o}(t^{1/3})\}\exp\{-t^{1/3}(v(c_0 - C_1\theta) + \pi^2/[2((c_0 + C_1\theta) - 2b\theta)^2])\}$$

which is $\mathrm{o}(\exp\{-t^{1/3}c(1,v)\})$ as $t\uparrow\infty$ for $\theta > 0$ small and $b > c_0^3 vC_1/(2\pi^2) + C_1/2$.

In the third line we applied Lemma 3.1 and the upperbound in Lemma 3.3 ($s=1$). In the last line, in evaluating the expectation, we put the combinatorial complexity of discretized positions of $\tau \subset I_M$ in the pre-factor; also, as there are no Poisson points in $J_{\theta,t,b}$, we bounded $\lambda_1^{J_{\theta,t,b},V} = \lambda_1^{J_{\theta,t,b}} \geqslant t^{-2/3}\pi^2/[2((c_0 + C_1\theta) - 2b\theta)^2]$.

This finishes the proof with $\varepsilon = (C_1 + C_2)\theta$, $\xi_0 = (C_1 + C_2 + b)/(C_1 + C_2)$, $t^{1/3}\mathbb{I}^0_{\varepsilon,\omega} = \tau$ and $t^{1/3}\mathbb{J}_{\varepsilon,0,\xi_0} = J_{\theta,t,b}$. $\square$

Proof of Proposition 2.1 ($0 < |h| < \beta_0(1)$, (A)). We will assume $0 < h < \beta_0(1)$ without loss of generality as we could work with the Brownian process $-X(\cdot)$ otherwise. Let $\theta > 0$ be small in comparison to c_h. Recall the estimate on the normalization of Q_t^h (1.5) for $0 < h < \beta_0(1)$, and also the large deviation result near (1.4). Here, "atypical" or "not typical" events are on the order $\mathrm{o}(\exp\{-t^{-1/3}c(1, v-h)\})$.

Step 1: The minimum of the rate function J is at c_h, so we restrict to those paths such that $t^{-1/3}X(t) \in L(c_h,\theta)$ where $L(c,\theta) = [c-\theta, c+\theta]$. On this set, the term $\exp\{hX(t)\} \leqslant \exp\{h(c_h + \theta)t^{1/3}\}$. Therefore

$$\mathbb{E}\otimes E_0\left[\exp\left\{hX(t) - \int_0^t V(X(r))\,\mathrm{d}r\right\}, t^{-1/3}X(t) \in L(c_h,\theta)\right]$$

$$\leqslant \exp\{h(c_h+\theta)t^{1/3}\}\cdot \mathbb{E}\otimes E_0\left[\exp\left\{-\int_0^t V\,\mathrm{d}r\right\}, X(t) \in t^{1/3}L(c_h,\theta)\right].$$

Step 2: We now follow the route of the previous proof for $h = 0$ to examine the $\mathbb{E}\otimes E_0$-expectation in the last line. On sets A defined near (4.5) for $y \in L(c_h,\theta)$ and $zt^{1/3} \in I_M$, we deduce that, typically, $|l_t - c_h| \leqslant C_0\theta$, and $|l_t' - c_h| \leqslant C_1\theta$, for some constants $1 \leqslant C_0 \leqslant C_1$, $C_0 = C_0(v,h,\beta_0(1))$, $C_1 = C_1(v,h,\beta_0(1))$.

Step 3: Define the exit time $S_{C,B}$ as before. The fact that typically $S_{C_4,\tau} > t$ for some large $C_4 = C_4(v,h,\beta_0(1))$ follows as in the proof for $h = 0$. Therefore, as both $\mathrm{dist}(0,\tau) \leqslant C_4\theta t^{1/3}$ and $\mathrm{dist}(X(t),\tau) \leqslant C_4\theta t^{1/3}$ for $X(t) \in t^{1/3}L(c_h,\theta)$, this locates $\tau \subset [-(C_1 + C_4 + 1)\theta, c_h + (C_1 + C_4)\theta]$.

This ends the argument, as for case $h = 0$, by picking ε appropriately. $\square$

Proof of Proposition 2.1 ($0 < |h| < \beta_0(1)$, (B)). We follow the first 2 steps of the previous part (A). If we now restrict the motion to the interval $J_{\theta,t,b} = \{x \in \tau\colon\ \mathrm{dist}(x, \partial\tau) > b\theta t^{1/3}\}$ for $b\theta < (c_h - C_1\theta)/2$, we have, analogously to the proof for $h = 0$, that

$$\mathbb{E} \otimes E_0 \left[A, \exp\left\{ -\int_0^t V\,\mathrm{d}r \right\}, |l_t - c_h| \leqslant C_0\theta, |l_t' - c_h| \leqslant C_1\theta, T_{J_{\theta,t,b}} > t \right]$$
$$\leqslant \exp\{\mathrm{o}(t^{1/3})\} \exp\{-t^{1/3}(v(c_h - C_1\theta) + \pi^2/[2((c_h + C_1\theta) - 2b\theta)^2])\}.$$

Now, to get bounds on $Q_t^h(T_{J_{\theta,t,b}} > t)$, we multiply this last term by the factor $\exp\{t^{1/3}h(c_h + \theta)\}$ (recall the term $\exp\{hX(t)\}$ in step 1, part (A)). The subsequent product is $\mathrm{o}(\exp\{-t^{1/3}c(1, v - h)\})$ as $t \uparrow \infty$ for large enough $b = b(v, h, c_h, C_1)$. This finishes the proof, as in the case $h = 0$. □

Proof of (4.5) and Proposition 4.1 (Outline). As remarked, the proofs of (4.5) and Proposition 4.1 follow from the proofs of Lemma 2.2 and Proposition 2.3 Povel, 1997. As the arguments there are long and intricate, we indicate here the main steps and their modifications which lead to the statement (4.7). Following the lead in Povel (1997), we discuss only the case $y > 0$ as the case $y \leqslant 0$ is similar.

The first step toward (4.5) is to modify the definition of the covering sets in Lemma 2.2 (Povel, 1997) to allow for $z \neq 0$. A covering set A in our context is also the union of several subsets, G_i for $i \leqslant 4$, G_5' and possibly an extra set $\tilde{G}_6$ if $yt^{1/3} > y_K + 2t^\delta$. The first four sets are the same as in Povel (1997): G_1 is the event that $\Theta_i \cap I_M$ for $1 \leqslant i \leqslant K$ given through (4.3) are pseudo-holes. G_2 is the event that the process returns exactly $J \leqslant [L(t)]$ times to pseudo-holes. G_3 is the set which specifies that the motion spends at least $(1 - \eta)t$ units of time in pseudo-holes. G_4 is the part where the path positions at all return times R_j for $1 \leqslant j \leqslant J$ lie in the pseudo-holes given by G_1. Here, more carefully, R_j specifies the jth return to the interior $\bigcup_i L_i$ after the $j - 1$th departure D_{j-1} from $\bigcup_i \Theta_i \cap I_M$. The event G_5' when $\min\{x_1 - 2t^\delta, 0\} \leqslant zt^{1/3} \leqslant \max\{yt^{1/3}, y_K + 2t^\delta\}$ (so that $zt^{1/3}$ is “between” visited pseudo-holes) is the same as the event G_5 in (Povel, 1997) which specifies that there exists a subsequence of the $\{D_j, R_{j+1}\}$ such that at times D_{j_i} and R_{j_i+1} the process is in different pseudo-holes and that every pseudo-hole in G_1 is visited by time t. When $zt^{1/3} < \min\{x_1 - 2t^\delta, 0\}$ or $zt^{1/3} > \max\{yt^{1/3}, y_K + 2t^\delta\}$ is to the left or right of all the pseudo-holes, G_5' changes so that $\mathscr{H}(z)$ can occur between a D_j and R_{j+1}:

$$G_5' = \{D_{\bar{j}} < \mathscr{H}(z) < R_{\bar{j}+1} \text{ for some } \bar{j} \leqslant J, \text{ and}$$
$$\exists \text{ subsequence } \{j_i\} \text{ of } \{1, \ldots, J\} \text{ and sequence } \{l_i\}, 1 \leqslant l_i \leqslant K$$
$$\text{s.t. } \bigcup_i \{l_i\} = \{1, \ldots, K\}, \text{ and } X(D_{j_i}) = u_{l_i},\ X(R_{j_i+1}) = v_{l_i}\}.$$

The sixth set $\tilde{G}_6$ is the same as in Povel (1997) and controls the behavior of the process after the last departure D_J if $y_K + 2t^\delta < yt^{1/3}$ so that the process reaches $yt^{1/3}$ before time t and returning to pseudo-holes.

The proof of (4.5) now follows almost the same scheme as for Lemma 2.2 (Povel, 1997).

To outline the proof of Proposition 4.1, we state some of the main steps in the proof of Proposition 2.3 (Povel, 1997) which lead to the result. In fact, with $z \in [\min\{x_1 - 2t^\delta, 0\}, \max\{yt^{1/3}, y_K + 2t^\delta\}]$, the proof follows easily from steps in (Povel, 1997): (2.70) the eigenvalue estimate, (2.78) the travel cost in the forest, and (2.86) the clearing cost.

More specifically, (2.70) gives that $\mathbb{E} \times E_0[A, \exp\{-\int_0^t V\,\mathrm{d}s\}]$, in terms of a small number $\rho_0 > 0$, is less than

$$\exp\{\mathrm{o}(t^{1/3})\}\mathbb{E}\left[G_1, \exp\{-(1-\rho_0)(1-\eta)t\lambda^{\bigcup_i \Theta_i \cap I_M, V}\},\right.$$
$$\left.\prod_{i=0}^{K-1} E_{u_i}\left[\exp\left\{-\int_0^{R_1} V\,\mathrm{d}s\right\},\ X(R_1) = v_i\right]\right] \tag{4.7}$$

with an extra term, $E_{y_K + t^\delta}[\exp\{-\int_0^{H(yt^{1/3})} V\,\mathrm{d}s\}, R_1 > H(yt^{1/3})]$, in the product if $yt^{1/3} > y_K + 2t^\delta$. Let B' denote the product in the $\mathbb{E}$ expectation.

From Lemma 3.3 ($s = 1$) and $l' \leqslant \gamma t^{1/3}$, we have

$$\lambda^{\bigcup_i \Theta_i \cap I_M, V} \geqslant \pi^2/(2(l')^2) + \mathrm{o}(t^{1/3}) \geqslant \pi^2/(2(\gamma t^{1/3})^2) + \mathrm{o}(t^{1/3})$$

(as $|\bigcup_i \Theta_i \cap I_M| > t^\delta$ grows large as $t \uparrow \infty$). This gives then

$$\mathbb{E} \times E_0\left[A, \exp\left\{-\int_0^t V\,\mathrm{d}s\right\}\right]$$
$$\leqslant \exp\{\mathrm{o}(t^{1/3})\}\exp\{-(1-\rho)(1-\eta)\pi^2/(2(\gamma t^{1/3})^2)\}\mathbb{E}[G_1, B'].$$

Let B'' be the $\mathbb{E}$ expectation term on the right. Then, from (2.77) and the end of Lemma 2.3 (Povel, 1997), we have B'' less than

$$\prod_{i=1}^{K} \mathbb{P}[\Theta_i \cap I_M \text{ contains at most } [2M/r] + 5 \text{ edges}] \cdot$$
$$\prod_{i=0}^{K-1} \mathbb{E} \times E_{u_i}\left[\exp\left\{-\int_0^{R_1} V\,\mathrm{d}s\right\}, X(R_1) = v_i\right]$$

with an additional factor in the second product corresponding to hitting $yt^{1/3}$ if $yt^{1/3} > y_K + 2t^\delta$. Let B_1 and B_2 denote the first and second products above.

Now, from (2.78) and (2.86) (Povel, 1997), we have

$$B_1 \leqslant \exp\{\mathrm{o}(t^{1/3})\}\exp\{-vC(\alpha_0; a)l\}$$

and

$$B_2 \leqslant \exp\{-\beta_0(1)(1-\varepsilon_0)\tilde{l}\}$$

where $\varepsilon_0 > 0$ is a small number and $C(\alpha_0; a) \to 1$ as $\alpha_0 \downarrow 0$. Therefore, from these observations, Proposition 4.1 follows in the case $z \in [\min\{x_1 - 2t^\delta, 0\}, \max\{yt^{1/3}, y_K + 2t^\delta\}]$.

The modifications for an A with $z \notin [\min\{x_1 - 2t^\delta, 0\}, \max\{yt^{1/3}, y_K + 2t^\delta\}]$ consist only of change in (2.70) (Povel, 1997) (or (4.7) above) B' to allow for the extra term with z replacing y when $\max\{y_K + 2t^\delta, yt^{1/3}\} \leqslant zt^{1/3}$ (if $zt^{1/3} < \min\{x_1 - 2t^\delta, 0\}$ then $u_0 = zt^{1/3}$ already reflects the change in this case). The proof of this change follows without much difficulty from Povel (1997) (2.63)–(2.69) pp. 1762–1763) by including a term in the surgery of the path there of the form

$$\exp\left\{-\int_{D_{\bar{j}}}^{H(zt^{1/3})} V\,\mathrm{d}s\right\}\exp\left\{-\int_{H(zt^{1/3})}^{R_{\bar{j}+1}} V\,\mathrm{d}s\right\}$$

if $\bar{j} < J$ or $\exp\{-\int_{D_{\bar{j}}}^{H(zt^{1/3})} V\,\mathrm{d}s\}$ if $\bar{j} = J$. The strong Markov property is then applied similarly. □

5. Proofs of the main theorems

Let $u \geqslant 0$ and let $f_u : \Omega \to \mathbb{R}$ be a bounded continuous function measurable with respect to $\mathscr{F}_u$. Let also $\mathbb{P}_s$ and $\mathbb{E}_s$, for $s \geqslant 1$, be the scaled point-process on $\mathbb{S}$ with intensity νs, and its expectation. It will be helpful to observe that if configurations ω are governed by $\mathbb{P}$, then the distribution of configurations ω_s is $\mathbb{P}_s$.

Define now

$$A^f_{s,h} = \mathbb{E}_s \otimes E_0\left[f(X(\cdot))\exp\left\{hsX(s) - \int_0^s \bar{V}_s(X(r),\omega)\,\mathrm{d}r\right\}\right]$$

(cf. definition of $\bar{V}_s$ (3.13)) and the functions $f^1(X(\cdot)) = f_u(X(\cdot))$ and $f^s(X(\cdot)) = f_u(sX(\cdot/s^2))$. With these definitions, we see from simple re-scalings that to prove the limits for $t^{-1/3}X(\cdot t^{2/3})$ and $X(\cdot)$ under $\mathrm{d}Q^h_t$, and therefore Theorems 2.1 and 2.2, is the same as to show the convergences

$$\lim_{t\to\infty} A^{f^1}_{t^{1/3},h}/A^1_{t^{1/3},h} = \begin{cases} E_{Q^{(-c_0/2,c_0/2)}}[f_u] & \text{for } h = 0 \\ E_{\mathrm{P}_0^{(0,c_h)}}[f_u] & \text{for } 0 < |h| < \beta_0(1) \end{cases}$$

and

$$\lim_{t\to\infty} A^{f^{t^{1/3}}}_{t^{1/3},0}/A^1_{t^{1/3},0} = E_0[f_u]$$

for all bounded continuous $f_u \in \mathscr{F}_u$ and $u \geqslant 0$.

Indeed, with respect to the "scaled" limit of $t^{1/3}X(\cdot t^{2/3})$ under $\mathrm{d}Q_t^h$ compute that

$$\begin{aligned}
&\mathbb{E}\otimes E_0\left[f_u(t^{-1/3}X(\cdot t^{2/3}))\exp\left\{hX(t)-\int_0^t V(X(r),\omega)\,\mathrm{d}r\right\}\right]\\
&\quad=\mathbb{E}\otimes E_0\left[f_u(X(\cdot))\exp\left\{ht^{1/3}X(t^{1/3})-\int_0^t V(t^{1/3}X(rt^{-2/3}),\omega)\,\mathrm{d}r\right\}\right]\\
&\quad=\mathbb{E}\otimes E_0\left[f^1,\exp\left\{ht^{1/3}X(t^{1/3})-\int_0^{t^{1/3}} t^{2/3}V(t^{1/3}X(r),\omega)\,\mathrm{d}r\right\}\right]\\
&\quad=\mathbb{E}_{t^{1/3}}\otimes E_0\left[f^1,\exp\left\{ht^{1/3}X(t^{1/3})-\int_0^{t^{1/3}} t^{2/3}V(t^{1/3}X(r),t^{1/3}\omega)\,\mathrm{d}r\right\}\right]\\
&\quad=A^{f^1}_{t^{1/3},h}. \qquad (5.1)
\end{aligned}$$

A similar calculation holds for the "unscaled" limit.

To simplify notation, denote $s=s(t)=t^{1/3}$ in the rest of the section. The strategy now will be to determine, through large deviation estimates, the leading order asymptotics of the terms $A^{f^1}_{s(t),h}$ and $A^{f^{s(t)}}_{s(t),h}$ in comparison to $A^1_{s(t),h}=\exp\{-s(t)(c(1,v-h)+\mathrm{o}(1))\}$ (cf. (1.5)). To simplify the presentation, we first prove the scaled and unscaled limits for the case $h=0$. Then we prove the scaled limit for the case $0<|h|<\beta_0(1)$ by making the necessary departures from the drift-free arguments.

5.1. *Proof of Theorem 2.1: $h=0$*

The proof follows in several steps where we isolate the dominant modes of $A^{f^1}_{s(t),0}$ and $A^{f^{s(t)}}_{s(t),0}$ in comparison to $A^1_{s(t),0}=\exp\{-s(t)(c(1,v)+\mathrm{o}(1))\}$. As the arguments for the scaled and unscaled limits are virtually the same until the last two steps, we concentrate on the limit for f^1 until the end.

Step 1: It will be useful to reformulate the problem on an interval. Let ζ be the constant in Proposition 2.1 and consider $0<\varepsilon<c_0/30$. Let $G_{\varepsilon,0}$ be the collection of configurations ω which contain a unique empty interval $I^0_\varepsilon=I^0_{\varepsilon,\omega}\subset[-c_0-\varepsilon,c_0+\varepsilon]$ with length $|I^0_\varepsilon|\in[c_0-\varepsilon,c_0+\varepsilon]$, and second largest empty interval of size less than $2\zeta\varepsilon$ in the interval $[-c_0-2\varepsilon,c_0+2\varepsilon]$, viewed as a torus with endpoints indentified (so the second largest empty interval could be the union of empty intervals on the ends). Let $B^0_\varepsilon=B^0_{\varepsilon,\omega}$ be the open ε-neighborhood of $I^0_{\varepsilon,\omega}$.

Then, on $G_{\varepsilon,0}$ we have that the length of the second largest interval in B^0_ε, empty of Poisson points, is bounded above by $2\varepsilon<(c_0-\varepsilon)/10\leqslant|I^0_\varepsilon|/10$. Define the event

$$F_{\varepsilon,\omega,0}=G_{\varepsilon,0}\cap\{T_{B^0_{\varepsilon,\omega}}>s(t)\}.$$

From scaling calculations similar to (5.1) and Proposition 2.1 (A), we have that

$$\mathbb{E}_{s(t)} \otimes E_0 \left[F^c_{\varepsilon,\omega,0}, f_u, \exp\left\{ -\int_0^{s(t)} \bar{V}_{s(t)}(X(r),\omega)\,\mathrm{d}r \right\} \right] = \mathrm{o}(A^1_{s(t),0}).$$

We may concentrate, therefore, on limits of the ratio

$$\mathbb{E}_s \otimes E_0 \left[F_{\varepsilon,\omega,0}, f_u, \exp\left\{ -\int_0^s \bar{V}_s(X(r),\omega)\,\mathrm{d}r \right\} \right] / A^1_{s,0}.$$

Step 2: We now decompose $A^{f^1}_{s,0}$ further.

Step 2.1: Due to the measurability of f_u on paths up to time u, we may write, by semi-group estimates, $A^{f^1}_{s,0} = L_1 + L_2 + \mathrm{o}(A^1_{s,0})$ where

$$L_1 = \mathbb{E}_s \otimes E_0 \Bigg[G_{\varepsilon,0}, T_{B^0_\varepsilon} > u, f_u, \exp\left\{ -\int_0^u \bar{V}_s(X(r))\,\mathrm{d}r \right\}, \phi_1^{B^0_\varepsilon,\bar{V}_s}(X(u)),$$

$$\langle 1, \phi_1^{B^0_\varepsilon,\bar{V}_s} \rangle_{L^2}, \exp\{-\lambda_1^{B^0_\varepsilon,\bar{V}_s}(s-u)\} \Bigg]$$

and

$$L_2 = \mathbb{E}_s \otimes E_0 \Bigg[G_{\varepsilon,0}, T_{B^0_\varepsilon} > u, f_u, \exp\left\{ -\int_0^u \bar{V}_s(X(r))\,\mathrm{d}r \right\},$$

$$[(R_{s-u}^{B^0_\varepsilon,\bar{V}_s} 1)(X(u)) - \phi_1^{B^0_\varepsilon,\bar{V}_s}(X(u)) \langle 1, \phi_1^{B^0_\varepsilon,\bar{V}_s} \rangle_{L^2} \exp\{-\lambda_1^{B^0_\varepsilon,\bar{V}_s}(s-u)\}] \Bigg].$$

Step 2.2: To bound L_2, we apply the semigroup estimate Lemma 3.1 and then Lemma 3.5 (with $U = sB^0_\varepsilon$, $r = (c_0 - \varepsilon)s \geqslant r_0$ for s large, and noting, in the definition of $\bar{V}_s$, that $s\omega$ appears in the second component of V_s and $(s\omega)_s = \omega$), to get, for large times, that

$$L_2 \leqslant C(f_u)\mathbb{E}_s[G_{\varepsilon,0}, \exp\{-\mathscr{C}\lambda_1^{B^0_\varepsilon,\omega}(s-u)\}],$$

where $\mathscr{C} > 1$. To bound the expectation further, note that by discretizing $\mathbb{R}$ into divisions of length s^{-3}, say, the number of possible locations for the random interval $I^0_\varepsilon \subset [-2c_0, 2c_0]$ is bounded as $\mathrm{O}(\mathrm{e}^{\mathrm{o}(s)})$ for large times. Therefore, for $0 < \varepsilon < (\mathscr{C} - 1)(\pi^2/(2c_0^2))/(v + \mathscr{C}\pi^2/c_0^3)$, the above expectation is bounded above by $\exp\{-s[v(c_0 - \varepsilon) + \mathscr{C}\pi^2/(2(c_0+\varepsilon)^2) + \mathrm{o}(1)]\} = \mathrm{o}(A^1_{s,0})$ for large times.

Step 3: We analyze L_1 and isolate the dominant term.

Step 3.1: By Lemmas 3.3 and 3.6, and (2.1) (with again $U = sB^0_\varepsilon$, noting that the "ω" in these lemmas take the form $s\omega$ here and $(s\omega)_s = \omega$, and for Lemma 3.6, $c = (c_0 - \varepsilon)/2$ and $d = c_0 + 3\varepsilon$), we have uniformly over $\omega \in G_{\varepsilon,0}$, for large enough s, that

$$|\lambda_1^{B^0_\varepsilon,\bar{V}_s} - \lambda_1^{B^0_\varepsilon,\omega}| \leqslant C(W, c_0)s^{-1},$$

$$|\lambda_1^{B^0_\varepsilon,\omega} - \pi^2/(2c_0^2)| \leqslant C(W, c_0)\varepsilon,$$

$$\|\phi_1^{B^0_\varepsilon,\bar{V}_s}\|_{L^\infty} \leqslant 2\sqrt{2/(c_0 - \varepsilon)}$$

and

$$\|\phi_1^{B_\varepsilon^0,\bar V_s} - \phi_1^{B_\varepsilon^0,\omega}\|_{L^\infty} \leqslant C_1(s; W, c_0, \varepsilon),$$

where $\lim_{s\uparrow\infty} C_1(s; W, c_0, \varepsilon) = 0$.

Step 3.2: To estimate L_1, we use the following trick of Sznitman (1991, cf. proof of Theorem 4.3). We replace the "soft" eigenfunction apparatus associated to $\phi_1^{I,\bar V_s}$ by the "hard obstacle" one corresponding to $\phi_1^{I,\omega}$. Let z_ω be the center of the critical clearing I_ε^0. Also let H_a be the interval $H_a = (-a/2, a/2)$. Write $L_1 = J_1 + J_2$ where

$$J_1 = \mathbb{E}_s \otimes E_0[G_{\varepsilon,0}, T_{H_{c_0}+z_\omega} > u, f_u, \phi_1^{H_{c_0}+z_\omega}(X(u)),$$
$$\exp\{-\lambda_1^{B_\varepsilon^0,\bar V_s}(s-u)\}, \langle 1, \phi_1^{B_\varepsilon^0,\bar V_s}\rangle_{L^2}],$$

$$J_2 = \mathbb{E}_s \left[G_{\varepsilon,0}, \exp\{-\lambda_1^{B_\varepsilon^0,\bar V_s}(s-u)\}, \langle 1, \phi_1^{B_\varepsilon^0,\bar V_s}\rangle_{L^2} \left\{E_0[f_u, \exp\left\{-\int_0^u \bar V_s\,\mathrm{d}r\right\},\right.\right.$$
$$\left.\left. T_{B_\varepsilon^0} > u, \phi_1^{B_\varepsilon^0,\bar V_s}(X(u))] - E_0[f_u, T_{H_{c_0}+z_\omega} > u, \phi_1^{H_{c_0}+z_\omega}(X(u))]\right\}\right].$$

Step 3.3: To bound the error J_2, we first bound

$$\left|E_0\left[f_u, \exp\left\{-\int_0^u \bar V_s\,\mathrm{d}r\right\}, T_{B_\varepsilon^0} > u, \phi_1^{B_\varepsilon^0,\bar V_s}(X(u))\right]\right.$$
$$\left. - E_0[f_u, T_{H_{c_0}+z_\omega} > u, \phi_1^{H_{c_0}+z_\omega}(X(u))]\right|$$

less than

$$\left|E_0\left[f_u, \exp\left\{-\int_0^u \bar V_s\,\mathrm{d}r\right\}, T_{B_\varepsilon^0} > u, \phi_1^{B_\varepsilon^0,\bar V_s}(X(u))\right]\right.$$
$$\left. - E_0[f_u, T_{H_{c_0-2\varepsilon}+z_\omega} > u, \phi_1^{H_{c_0-2\varepsilon}+z_\omega}(X(u))]\right|$$
$$+ |E_0[f_u, T_{H_{c_0}+z_\omega} > u, \phi_1^{H_{c_0}+z_\omega}(X(u))]$$
$$- E_0[f_u, T_{H_{c_0-2\varepsilon}+z_\omega} > u, \phi_1^{H_{c_0-2\varepsilon}+z_\omega}(X(u))]|.$$

We now estimate the first term by the method of Sznitman (1991, p. 1165) (cf. proof of Lemma 3.7 also). For $\omega \in G_{\varepsilon,0}$ and s large, we have $\bar V_s = 0$ on $H_{c_0-2\varepsilon} + z_\omega \subset B_\varepsilon^0$, and so $\mathrm{e}^{-\int_0^u \bar V_s(X(r),\omega)\,\mathrm{d}r} I(T_{B_\varepsilon^0} > u) \geqslant I(T_{H_{c_0-2\varepsilon}+z_\omega} > u)$. The first term is bounded now by

$$\|f_u\|_{L^\infty}(E_0[(\mathrm{e}^{-\int_0^u \bar V_s\,\mathrm{d}r} I(T_{B_\varepsilon^0} > u) - I(T_{H_{c_0-2\varepsilon}+z_\omega} > u))\phi_1^{B_\varepsilon^0,\bar V_s}(X(u))]$$
$$+ E_0[I(T_{H_{c_0-2\varepsilon}+z_\omega} > u)|\phi_1^{B_\varepsilon^0,\bar V_s} - \phi_1^{H_{c_0-2\varepsilon}+z_\omega}|(X(u))])$$

$$\leqslant \|f_u\|_{L^\infty}(2\|\phi_1^{B_\varepsilon^0,\bar V_s} - \phi_1^{H_{c_0-2\varepsilon+z_\omega}}\|_{L^\infty} + |E_0[\mathrm{e}^{-\int_0^u \bar V_s\,\mathrm{d}r} I(T_{B_\varepsilon^0} > u)\phi_1^{B_\varepsilon^0,\bar V_s}(X(u))]$$

$$-E_0[I(T_{H_{c_0-2\varepsilon+z_\omega}} > u)\phi_1^{H_{c_0-2\varepsilon+z_\omega}}(X(u))]])$$

$$\leqslant \|f_u\|_{L^\infty}(2\|\phi_1^{B_\varepsilon^0,\bar V_s} - \phi_1^{H_{c_0-2\varepsilon+z_\omega}}\|_{L^\infty}$$

$$+|\exp\{-\lambda_1^{B_\varepsilon^0,\bar V_s}u\}\phi_1^{B_\varepsilon^0,\bar V_s}(0) - \exp\{-\lambda_1^{H_{c_0-2\varepsilon+z_\omega}}u\}\phi_1^{H_{c_0-2\varepsilon+z_\omega}}(0)|).$$

This is further bounded, using the bounds in substep 3.1 for large times, by

$$\|f_u\|_{L^\infty}(C(s;W,c_0) + \|\phi_1^{B_\varepsilon^0,\omega} - \phi_1^{H_{c_0-2\varepsilon+z_\omega}}\|_{L^\infty}$$

$$+|\exp\{-\lambda_1^{B_\varepsilon^0,\omega}u\} - \exp\{-\lambda_1^{H_{c_0-2\varepsilon+z_\omega}}u\}|)$$

where $C(s;W,c_0)\downarrow 0$ as $s\uparrow\infty$. By explicit computation with Dirichlet eigenvalues and eigenfunctions, using (2.1) and $\||I_\varepsilon^0| - c_0| \leqslant \varepsilon$, we bound this term further by $C(s,\varepsilon) = C(s,\varepsilon; f_u, W, c_0)$ uniformly over $\omega\in G_{\varepsilon,0}$ where $\lim_{\varepsilon\downarrow 0}\lim_{s\uparrow\infty} C(s,\varepsilon) = 0$.

The second term is bounded similarly with the same bound. All this gives

$$|J_2| \leqslant 2C(s,\varepsilon)\mathbb{E}_s[G_{\varepsilon,0}, \exp\{-\lambda_1^{B_\varepsilon^0,\bar V_s}(s-u))\}, \langle 1, \phi_1^{B_\varepsilon^0,\bar V_s}\rangle_{L^2}].$$

Step 3.4: It will be convenient to bound J_2 further. Observe on the set $G_{\varepsilon,0}$ that z_ω belongs exactly to $H_{c_0+3\varepsilon}$, that is $I(G_{\varepsilon,0}) = I(G_{\varepsilon,0}, z_\omega\in H_{c_0+3\varepsilon})$. Also, when $z_\omega\in[-c_0/2-3\varepsilon/2, -c_0/2-\varepsilon/2)\cup(c_0/2+\varepsilon/2, c_0/2+3\varepsilon]$, the full range of lengths of $I_{\varepsilon,\omega}^0$ are not possible and in fact $|I_{\varepsilon,\omega}^0| \leqslant 2\|z_\omega| - c_0 - \varepsilon| < c_0+\varepsilon$. In contrast, when $z_\omega\in H_{c_0+\varepsilon}$, all lengths $|I_{\varepsilon,\omega}^0|\in[c_0-\varepsilon, c_0+\varepsilon]$ are possible. Also, we observe that $I(G_{\varepsilon,0})\exp\{-\lambda_1^{B_{\varepsilon,\omega}^0,\bar V_s(\cdot,\omega)}(s-u)\}\langle 1, \phi_1^{B_{\varepsilon,\omega}^0,\bar V_s(\cdot,\omega)}\rangle_{L^2}$ and $\mathbb{P}_s$ are invariant to rotations of configurations ω on the interval $[-c_0-2\varepsilon, c_0+2\varepsilon]$ thought of as a torus with ends indentified where the rotation keeps $z_\cdot\in H_{c_0+3\varepsilon}$. Therefore, we conclude that

$$\mathbb{E}_s[G_{\varepsilon,0}, \exp\{-\lambda_1^{B_\varepsilon^0,\bar V_s}(s-u))\}, \langle 1, \phi_1^{B_\varepsilon^0,\bar V_s}\rangle_{L^2}]$$

$$\leqslant \left(1+\frac{3\varepsilon}{c_0}\right)\mathbb{E}_s[G_{\varepsilon,0}, z_\omega\in H_{c_0}, \exp\{-\lambda_1^{B_\varepsilon^0,\bar V_s}(s-u))\}, \langle 1, \phi_1^{B_\varepsilon^0,\bar V_s}\rangle_{L^2}].$$

And so, finally,

$$|J_2| \leqslant 2C(s,\varepsilon)\left(1+\frac{3\varepsilon}{c_0}\right)\mathbb{E}_s[G_{\varepsilon,0}, z_\omega\in H_{c_0}, \exp\{-\lambda_1^{B_\varepsilon^0,\bar V_s}(s-u))\}, \langle 1, \phi_1^{B_\varepsilon^0,\bar V_s}\rangle_{L^2}]. \tag{5.2}$$

Step 4: We now finish the proof of the case $h=0$.

Step 4.1: Rewrite, from calculation, that J_1 equals

$$\mathbb{E}_s[G_{\varepsilon,0}, \mathrm{e}^{-\lambda_1^{B_\varepsilon^0,\bar V_s}(s-u)}, \langle 1, \phi_1^{B_\varepsilon^0,\bar V_s}\rangle_{L^2},\ E_{-z_\omega}[T_{H_{c_0}} > u, f_u(X(\cdot)+z_\omega), \phi_1^{H_{c_0}}(X(u))]].$$

Observe, now, that the Brownian expectation, $\Psi(z) = E_z[T_{H_{c_0}} > u, f_u(X(\cdot) - z), \phi_1^{H_{c_0}}(X(u))]$, vanishes for $|z| \geqslant c_0/2$ and is uniformly continuous on $[-c_0/2, c_0/2]$. Indeed, we have $|\Psi(z) - \Psi(z+\theta)|$ less than

$$E_z[f_u(X(\cdot) - z)|I(T_{H_{c_0}} > u)\phi_1^{H_{c_0}}(X(u)) - I(T_{H_{c_0}-\theta} > u)\phi_1^{H_{c_0}-\theta}(X(u))|]$$

which is less than (following substep 3.3)

$$\begin{aligned}
&\|f\|_{L^\infty}\{E_z[|I(T_{H_{c_0}} > u)\phi_1^{H_{c_0}}(X(u)) - I(T_{H_{c_0-2\theta}} > u)\phi_1^{H_{c_0-2\theta}}(X(u))|] \\
&\quad + E_z[|I(T_{H_{c_0}-\theta} > u)\phi_1^{H_{c_0}-\theta}(X(u)) - I(T_{H_{c_0-2\theta}} > u)\phi_1^{H_{c_0-2\theta}}(X(u))|]\} \\
&\leqslant \|f\|_{L^\infty}\{2\|\phi_1^{H_{c_0}} - \phi_1^{H_{c_0-2\theta}}\|_{L^\infty} + \mathrm{e}^{-\pi^2 u/(2c_0^2)}\phi_1^{H_{c_0}}(z) - \mathrm{e}^{-\pi^2 u/(2(c_0-2\theta)^2)} \\
&\quad \times \phi_1^{H_{c_0-2\theta}}(z) + 2\|\phi_1^{H_{c_0}-\theta} - \phi_1^{H_{c_0-2\theta}}\|_{L^\infty} + \mathrm{e}^{-\pi^2 u/(2c_0^2)}\phi_1^{H_{c_0}-\theta}(z) \\
&\quad - \mathrm{e}^{-\pi^2 u/(2(c_0-2\theta)^2)}\phi_1^{H_{c_0-2\theta}}(z)\}.
\end{aligned}$$

This last expression, noting (2.1), is $\mathrm{O}(\theta)$ uniformly for $z \in [-c_0/2, c_0/2]$ as $\theta \downarrow 0$.

Dividing now, according to the possible values of $z_\omega \in H_{c_0+3\varepsilon}$, we write J_1 further as

$$\begin{aligned}
&\sum_{k=1}^{n} \mathbb{E}_s[G_{\varepsilon,0}, \exp\{-\lambda_1^{B_\varepsilon^0,\bar{V}_s}(s-u)\}, \langle 1, \phi_1^{B_\varepsilon^0,\bar{V}_s}\rangle, \\
&\quad z_\omega \in [-c_0/2 + (k-1)c_0/n, -c_0/2 + kc_0/n]] \cdot \\
&\quad (E_{-(-c_0/2+kc_0/n)}[T_{H_{c_0}} > u, f_u(X(\cdot) + c_0/2 + kc_0/n), \phi_1^{H_{c_0}}(X(u))] + \delta(n)) \\
&\quad + \mathbb{E}_s[G_{\varepsilon,0}, \exp\{-\lambda_1^{B_\varepsilon^0,\bar{V}_s}(s-u)\}, z_\omega \in M(c_0,\varepsilon)] \cdot \\
&\quad E - z_\omega[T_{H_{c_0}} > u, f_u(X(\cdot) + z_\omega), \phi_1^{H_{c_0}}(X(u))]],
\end{aligned}$$

where $\delta(n)$ is the modulus of continuity of Ψ with $\lim_{n\uparrow\infty}\delta(n) = 0$ and $M(c_0,\varepsilon) = [-c_0/2 - 3\varepsilon/2, -c_0/2] \cup [c_0/2, c_0/2 + 3\varepsilon/2]$. The last term corresponding to $M(c_0,\varepsilon)$ vanishes as $|-z_\omega| \geqslant c_0/2$. Now, analogous to the estimation of J_2 in step 3.4, by rotation invariance of $I(G_{\varepsilon,0})\exp\{-\lambda_1^{B_\varepsilon^0,\bar{V}_s}(s-u)\}\langle 1, \phi_1^{B_\varepsilon^0,\bar{V}_s}\rangle_{L^2}$ and $\mathbb{P}_s$ with respect to shifts of configurations on $[-c_0 - 2\varepsilon, c_0 + 2\varepsilon]$, seen as a torus with ends identified, which keeps $z_\cdot \in H_{c_0}$, we have that the Poisson expectation in the sum equals

$$\begin{aligned}
&\mathbb{E}_s[\ldots, z_\omega \in [-c_0/2 + (k-1)c_0/n, -c_0/2 + kc_0/n]] \\
&\quad = \frac{1}{n}\mathbb{E}_s[\ldots, z_\omega \in [-c_0/2, c_0/2]].
\end{aligned}$$

We obtain now that J_1 converges as $n\uparrow\infty$ to an expression with split Brownian and Poisson factors,

$$\mathbb{E}_s[G_{\varepsilon,0}, z_\omega\in H_{c_0}, \exp\{-\lambda_1^{B^0_\varepsilon,\bar V_s}(s-u)\}, \langle 1,\phi_1^{B^0_\varepsilon,\bar V_s}\rangle_{L^2}]\cdot$$
$$\frac{1}{c_0}\int_{-c_0/2}^{c_0/2}\mathrm{d}x E_x[T_{H_{c_0}}>u, f_u(X(\cdot)-x), \phi_1^{H_{c_0}}(X(u))].$$

Note, if the function f_u is the constant 1, then the above reduces to

$$\mathbb{E}[\cdots]\cdot\frac{1}{c_0}\int_{-c_0/2}^{c_0/2}\mathrm{d}x\phi_1^{H_{c_0}}(x)\mathrm{e}^{-\pi^2u/(2c_0^2)}.$$

Step 4.2: Similarly, for the scaled limit, we get $A_s^{f^s}=J_1+J_2+\mathrm{o}(A_s^1)$ where J_1 equals

$$\mathbb{E}_s[G_{\varepsilon,0}, z_\omega\in H_{c_0}, \exp\{-\lambda_1^{B^0_\varepsilon,\bar V_s}(s-u/s^2)\}, \langle 1,\phi_1^{B^0_\varepsilon,\bar V_s}\rangle_{L^2}]\cdot$$
$$\frac{1}{c_0}\int_{-c_0/2}^{c_0/2}\mathrm{d}x E_x[f_u(sX(\cdot/s^2)-sx), \phi_1^{H_{c_0}}(X(u/s^2)), T_{H_{c_0}}>u/s^2]$$

and J_2 satisfies (5.2). Observing now that $s(X(\cdot/s^2)-X(0))\equiv(X(\cdot)-X(0))$ in law, we rewrite J_1 as

$$\mathbb{E}_s[G_{\varepsilon,0}, z_\omega\in H_{c_0}, \exp\{-\lambda_1^{B^0_\varepsilon,\bar V_s}(s-u/s^2)\}, \langle 1,\phi_1^{B^0_\varepsilon,\bar V_s}\rangle_{L^2}]]\cdot$$
$$E_0\left[f_u(X(\cdot))\frac{1}{c_0}\int_{-c_0/2}^{c_0/2}\mathrm{d}xI(T_{s(H_{c_0}-x)}>u)\phi_1^{H_{c_0}}(X(u)/s+x)\right].$$

As $\|\phi_1^{H_{c_0}}\|_{L^\infty}<\sqrt{2/c_0}$ we have that $\int_{-c_0/2}^{c_0/2}\mathrm{d}xI(T_{s(H_{c_0}-x)}>u)\phi_1^{H_{c_0}}(X(u)/s+x)$ converges to $\int_{-c_0/2}^{c_0/2}\mathrm{d}x\phi_1^{H_{c_0}}(x)$, as $s\uparrow\infty$, a.s. (P_0). Together, these observations imply, as $s\uparrow\infty$, that

$$E_0\left[f_u(X(\cdot)), \frac{1}{c_0}\int_{-c_0/2}^{c_0/2}\mathrm{d}x1_{(T_{s(H_{c_0}-x)}>u)}, \phi_1^{H_{c_0}}(X(u)/s+x)\right]$$
$$\to E_0[f_u]\langle 1,\phi_1^{H_{c_0}}\rangle_{L^2}/c_0.$$

Step 4.3: At this point, we note that the term $A^1_{s,0}$ may be decomposed exactly as we have done for $A^{f^1}_{s,0}$ and $A^{f^s}_{s,0}$ by taking the function $f\equiv 1$. Then, in the decomposition of $A^1_{s,0}$, we also obtain a dominant term where the Poisson and Brownian expectations decouple. In fact, the Poisson expectation factors for $A^1_{s,0}$, and $A^{f^1}_{s,0}$ and $A^{f^s}_{s,0}$ are the same and so these will cancel in the ratios $A^{f^1}_{s,0}/A^1_{s,0}$ and $A^{f^s}_{s,0}/A^1_{s,0}$.

Therefore, taking account of (5.2), the conclusions in steps 4.1 and 4.2, and the last paragraph, we have that

$$\overline{\lim_{\varepsilon\downarrow 0}}\ \overline{\lim_{t\uparrow\infty}}\ |A^{f^1}_{t^{1/3},0}/A^1_{t^{1/3},0} - E_{Q^{(-c_0/2,c_0/2)}}[f_u]| = 0$$

and

$$\overline{\lim_{\varepsilon\downarrow 0}}\ \overline{\lim_{t\uparrow\infty}}|A^{f^{t^{1/3}}}_{t^{1/3},0}/A^1_{t^{1/3},0} - E_0[f_u]| = 0.$$

This finishes the proof for the case $h = 0$. $\square$

5.2. *Proof of Theorem 2.2:* $0 < |h| < \beta_0(1)$

Without loss of generality, we assume that $0 < h < \beta_0(1)$, as we could consider the Brownian process $-X(\cdot)$ just as well. We will follow the same framework, as for the drift $h = 0$ case, to reduce $A^{f^1}_{s(t),h}$ to dominant terms, in comparison to $A^1_{s(t),h} = \exp\{-s(t)(c(1, v-h) + \mathrm{o}(1))\}$.

Step 1: As before, we place the problem on an interval. Recall Proposition 2.1 (A) when $0 < |h| < \beta_0(1)$. For $\varepsilon \in (0, c_h/100)$, let $G_{\varepsilon,h}$ be the set of configurations ω which have an empty interval $I^h_\varepsilon = I^h_{\varepsilon,\omega} \subset [-\varepsilon, c_h + \varepsilon]$ with length $|I^h_{\varepsilon,\omega}| \in [c_h - \varepsilon, c_h + \varepsilon]$. Let $B^h_\varepsilon = B^h_{\varepsilon,\omega}$ be the open ε-neighborhood of $I^h_{\varepsilon,\omega}$, and set $\mathscr{B}^h_\varepsilon = (-4\varepsilon, c_h + 4\varepsilon)$. On $G_{\varepsilon,h}$, the length of the second-largest empty subinterval in $\mathscr{B}^h_\varepsilon$ is less than $9\varepsilon < (c_h - \varepsilon)/10 \leqslant |I^h_\varepsilon|/10$. Define, analogously to the previous subsection, the event

$$F_{\varepsilon,\omega,h} = G_{\varepsilon,h} \cap (T_{\mathscr{B}^h_\varepsilon} > s(t)).$$

As, $B^h_\varepsilon \subset \mathscr{B}^h_\varepsilon$, scaling arguments analogous to (5.1) and Proposition 2.1 (A) imply that

$$\mathbb{E}_{s(t)} \otimes E_0 \left[F^c_{\varepsilon,\omega,h}, f_u, \exp\left\{ hs(t)X(s(t)) - \int_0^{s(t)} \bar{V}_{s(t)}(X(r),\omega)\,\mathrm{d}r \right\}\right] = \mathrm{o}(A^1_{s(t),h}).$$

Analogous to the $h = 0$ case, we may concentrate, therefore, on limits of the ratio

$$\mathbb{E}_s \otimes E_0 \left[F_{\varepsilon,\omega,h}, f_u, \exp\left\{ hsX(s) - \int_0^s \bar{V}_s(X(s),\omega)\,\mathrm{d}s \right\}\right] /A^1_{s,h}.$$

Step 2: As for the case $h = 0$ (step 2), we may decompose $A^{f^1}_{s,h}$ by eigenfunction expansion. We obtain that $A^{f^1}_{s,h}$ equals

$$\mathbb{E}_s \otimes E_0 \left[G_{\varepsilon,h}, T_{\mathscr{B}^h_\varepsilon} > u, f_u(X(\cdot)), \exp\left\{ -\int_0^u \bar{V}_s(X(r),\omega)\,\mathrm{d}r \right\},\right.$$
$$\left. \phi_1^{\mathscr{B}^h_\varepsilon,\bar{V}_s}(X(u)), \langle \exp\{hs\cdot\}, \phi_1^{\mathscr{B}^h_\varepsilon,\bar{V}_s}(\cdot)\rangle_{L^2}, \exp\{-\lambda_1^{\mathscr{B}^h_\varepsilon,\bar{V}_s}(s-u)\}\right] + \mathrm{o}(A^1_{s,h}). \qquad (5.3)$$

Step 3: Recall the definition of the taboo measure $\bar{P}^I_{\omega,x,s}$ and expectation $\bar{E}^I_{\omega,s,x}$ in (3.14).

Step 3.1: With I in the definition taken to be $I = \mathscr{B}^h_\varepsilon$, we now rewrite (5.3) as

$$\mathbb{E}_s[G_{\varepsilon,h}, \exp\{-\lambda_1^{\mathscr{B}^h_\varepsilon, \bar{V}_s} s\},$$
$$\langle \exp\{hs\cdot\}, \phi_1^{\mathscr{B}^h_\varepsilon, \bar{V}_s}(\cdot)\rangle_{L^2}, \phi_1^{\mathscr{B}^h_\varepsilon, \bar{V}_s}(0), \bar{E}^{\mathscr{B}^h_\varepsilon}_{\omega,0,s}[f_u(X(\cdot))]] + \mathrm{o}(A^1_{s,h}). \tag{5.4}$$

Step 3.2: The idea now is to replace $\bar{E}^{\mathscr{B}^h_\varepsilon}_{\omega,0,s}[f_u(X(\cdot))]$ for $\omega \in G_{\varepsilon,h}$ by the constant $E_{\mathrm{P}_0^{(0,c_h)}}[f_u]$ plus a uniform error. By Proposition 3.2, with $\bar{a} = 4\varepsilon$, $\bar{b} = c_h + 4\varepsilon$, $\bar{c} = c_h$, and $\beta = \varepsilon$, for ε small, (cf. Fig. 2 in Section 3.3) we have that (5.4) becomes

$$\mathbb{E}_s[\ldots, (E_{\mathrm{P}_0^{(0,c_h)}}[f_u] + \mathrm{error})] + \mathrm{o}(A^1_{s,h}),$$

where uniformly for $\omega \in G_{\varepsilon,h}$ the $|\mathrm{error}| \leqslant C(s,\varepsilon)$ and $\lim_{\varepsilon\downarrow 0}\lim_{s\uparrow\infty} C(s,\varepsilon) = 0$. As for the case $h = 0$, after decomposing $A^1_{t^{1/3},h}$ and pulling out the constant $E_{\mathrm{P}_0^{(0,c_h)}}[f_u]$, the Poisson expectation terms cancel in the fraction $A^{f^1}_{t^{1/3},h}/A^1_{t^{1/3},h}$. This finishes the proof of Theorem 2.2 as all errors vanish by taking first $t \uparrow \infty$ and then $\varepsilon \downarrow 0$. $\square$

Acknowledgements

I would like to thank A.-S. Sznitman for his encouragement with this problem. Thanks are also due to T. Povel and J. Sethuraman for helpful discussions. Particularly, I am grateful to the referee for much constructive criticism and for pointing out many errors and inconsistencies.

References

Donsker, M., Varadhan, S.R.S., 1975. Asymptotics for the Wiener sausage. Comm. Pure Appl. Math. 28, 525–565.

Eisele, T., Lang, R., 1987. Asymptotics for the Wiener sausage with drift. Probab. Theory Related Fields 74, 125–140.

Gilbarg, D., Trudinger, N., 1994. Elliptic Partial Differential Equations of Second Order. Springer, Berlin, New York.

Karatzas, I., Shreve, S., 1991. Brownian Motion and Stochastic Calculus, 2nd Edition. Springer, Berlin, New York.

Knight, F.B., 1969. Brownian local times and taboo processes. Trans. Amer. Math. Soc. 143, 173–185.

Povel, T., 1995. On weak convergence of conditional survival measure of one dimensional Brownian motion with drift. Ann. Appl. Probab. 5, 222–238.

Povel, T., 1997. Critical large deviations of one dimensional annealed Brownian motion in a Poissonian potential. Ann. Probab. 25, 1735–1773.

Povel, T., 1998. The one-dimensional annealed δ-Lyapounov exponent. Ann. I.H.P. Probab. et Stat. 34, 61–72.

Povel, T., 1999. Confinement of Brownian motion among Poissonian obstacles in R^d, $d \geqslant 3$. Probab. Theory Related Fields 114, 177–205.

Schmock, U., 1990. Convergence of the normalized one dimensional Wiener sausage path measure to a mixture of Brownian taboo processes. Stochast. Stochast. Rep. 29, 171–183.

Sznitman, A.S., 1991. On the confinement property of two dimensional Brownian motion among Poissonian obstacles. Comm. Pure Appl. Math. 44, 1137–1170.

Sznitman, A.S., 1992. Brownian motion and obstacles. Proceedings of the First European Congress of Math. Birkhauser, Betsal.

Sznitman, A.S., 1995a. Annealed Lyapounov exponents and large deviations in a Poissonian potential I, II. Ann. Scient. Ec. Norm. (Suppl.) 4e, t.28, 345–390.

Sznitman, A.S., 1995b. Some aspects of Brownian motion in a Poissonian potential. In: Cranston, M., Pinsky, M. (Eds.), Stochastic Analysis, Proceedings of the Symposium on Pure Math., Amer. Math. Soc., Vol. 57, Providence, RI, Ithaca, NY, 1993, pp. 137–143.

Sznitman, A.S., 1999. Brownian Motion, Obstacles and Random Media. Springer, Berlin, New York.