REMARKS ON THE RANGE AND MULTIPLE RANGE OF RANDOM WALK UP TO THE TIME OF EXIT

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ABSTRACT. We consider the scaling behavior of the range and p-multiple range, that is the number of points visited and the number of points visited exactly $p \geq 1$ times, of simple random walk on $\mathbb{Z}^d$, for dimensions $d \geq 2$, up to time of exit from a domain D_N of the form $D_N = ND$ where $D \subset \mathbb{R}^d$, as $N \uparrow \infty$. Recent papers have discussed connections of the range and related statistics with the Gaussian free field, identifying in particular that the distributional scaling limit for the range, in the case D is a cube in $d \geq 3$, is proportional to the exit time of Brownian motion. The purpose of this note is to give a concise, different argument that the scaled range and multiple range, in a general setting in $d \geq 2$, both weakly converge to proportional exit times of Brownian motion from D, and that the corresponding limit moments are 'polyharmonic', solving a hierarchy of Poisson equations.

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1. Introduction and results

Let $\{X_n : n \geq 0\}$ be a simple random walk on $\mathbb{Z}^d$: That is,

$$\mathbb{P}\big(X_{n+1} = \mathbf{x} \pm \mathbf{e}_i | X_n = \mathbf{x}\big) = \frac{1}{2d},$$

for $1 \leq i \leq d$ where $\big\{\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0) : 1 \leq i \leq d\big\}$ is the standard basis of $\mathbb{Z}^d$. The range of random walk up to time n, denoted by $\mathcal{R}_n$, is the number of distinct sites visited up to time n. Correspondingly, for integers $p \geq 1$, the p-multiple range of random walk up to time n, denoted by $\mathcal{R}_n^{(p)}$, is the number of sites visited exactly p times up to time n.

To contrast with the range, the multiple range is a more delicate object. For instance, whereas the set visited by the random walk up to time n is connected and this range $\mathcal{R}_n$ is monotone in n, the analogous properties are not true for the p-multiple range set $\mathcal{R}_n^{(p)}$ that is visited exactly p times.

Although there is a large literature on the range and multiple range of random walk, often motivated by its applications and connections with other fields (cf. [3], [5], [8], [9], [11], [15], and references therein), the range and multiple range subject to constraints are less studied. In particular, the range and multiple range up to the exit time from a domain have natural interpretations, which depend interestingly on the starting point of the random walk and the shape of the domain.

Recently, a body of work (see [1], [12], [13]) has considered extremes and local times of random walk in such settings and connections with the Gaussian free field. We mention that [12], [13] consider scaling limits of the number of 'thick' points of continuous time random walk, starting at the origin, before its exit from a domain in $d \geq 2$. Thick points are those with visitation at least of order $a(\log N)^2$ in $d = 2$ or of order $a \log N$ in $d \geq 3$, where $a \geq 0$ and N is the length scale. In particular, when $a = 0$ and the domain is a cube in $d \geq 3$, [13, Theorem 1.5] shows that the scaled range R_N/N^2 converges weakly to a constant times the exit time of Brownian motion.

In this context, the purpose of this note is to present a concise, different argument in a more general setting (Theorems 1.1 and 1.2) that the distributional limits of the scaled range and multiple range up to the exit time from a domain in $d \geq 2$ connect to the exit times of Brownian motion and that their moments, as functions of the starting point, are 'polyharmonic' in that they solve a hierarchy

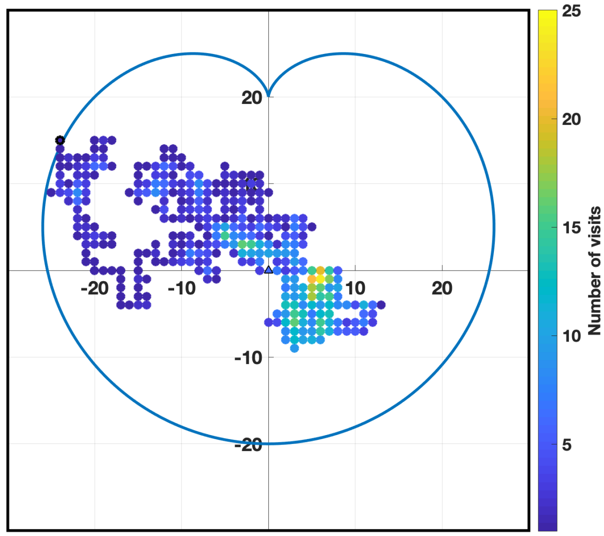

FIGURE 1. Range $\mathcal{R}_{\tau_N}$ and p-multirange $\mathcal{R}_n^{(p)}$ for a simple random walk on $\mathbb{Z}^2$, starting at the origin, up to the exit time from a cardioid D. For this realization, the exit time $n = 936$ and $\mathcal{R}_n^{(p)} = 0$ unless $1 \le p \le 24$.

of Poisson PDEs. In dimension $d = 2$, the scalings are different, involving extra '$\log N$' factors, than in $d \ge 3$. We also comment that the p-multiple range considers points with exactly p visitations, where p is fixed independent of N, a much different, and complementary object than the number of 'thick' points considered previously.

Here, the proofs of Theorems 1.1 and 1.2 make use of the known scaling behavior of the unconstrained range and multiple range, the functional central limit theorem for random walks, and moment bounds that we provide in this note through simple discrete Fourier analysis. Moreover, the argument shows that the randomness in the limits arises from the variability of the time to exit. In particular, we observe that these moment bound derivations, in terms of a notion of 'conductance', give a short way to see the scalings in Theorems 1.1 and 1.2.

We note that the phenomenon in $d \ge 2$ is different than in dimension $d = 1$, considered before in [4] for the range, where the ordering of space forces a different type of limit (cf. Remark 1.4). The methods in [4] in $d = 1$, make use of the reduced geometry, and do not carry over to higher dimensions $d \ge 2$.

Let $N \ge 1$ be a scaling parameter, and let $D_N = ND$ where $D \subset \mathbb{R}^d$ is an open, bounded domain with some regularity. To be definite, we will suppose that D is a Lipschitz domain, although in $d = 2$ it may be taken as a Jordan domain with a rectifiable boundary (which includes the case that D is Lipschitz). With respect to values $N \ge 1$, consider an array of simple random walks $\{X_n^{(N)} : n \ge 0\}$, starting from points $\{\mathbf{a}_N\} \subset D_N$, that is $X_0^{(N)} = \mathbf{a}_N$ for $N \ge 1$. We will assume that $\{\mathbf{a}_N\}$ satisfies $\mathbf{a}_N/N \to \mathbf{a}$ for some $\mathbf{a} \in D$.

Let now τ_N be the exit time from D_N by the random walk $\{X_n^{(N)}\}$, that is

$$\tau_N = \inf\left\{n \ge 0 : X_n^{(N)} \notin D_N\right\}.$$

In addition, for $r \ge 1$, let $T_{N,\mathbf{x}}^{(r)}$ be the rth hitting time of $\mathbf{x} \in \mathbb{Z}^d$. When $r = 1$, we will denote $T_{N,x} = T_{N,x}^{(1)}$. Define also $\tau_{\mathbf{a},D}$ as the exit time from D of a d-dimensional Brownian motion, starting from $\mathbf{a} \in D$.

With this notation, the range of the random walk $X_\cdot^{(N)}$ up to the time of exit is given by

$$R_N \;=\; \mathcal{R}_{\tau_N} \;=\; \sum_{\mathbf{x}\in D_N} 1\big(T_{N,\mathbf{x}} < \tau_N\big)$$

and the associated p-multiple range is given by

$$R_N^{(p)} \;=\; \mathcal{R}_{\tau_N}^{(p)} \;=\; \sum_{\mathbf{x}\in D_N} 1\big(T_{N,\mathbf{x}}^{(p)} < \tau_N < T_{N,\mathbf{x}}^{(p+1)}\big).$$

Finally, when $d \geq 3$, let $p_0 < 1$ be the probability that (transient) simple random walk on $\mathbb{Z}^d$, starting from the origin, returns.

Theorem 1.1 (Range)**.** *We have the weak convergence, as $N \uparrow \infty$, that*

$$\frac{R_N}{N^2/\log N} \Rightarrow \pi\tau_{\mathbf{a},D} \quad \text{when } d = 2, \quad \text{and}$$
$$\frac{R_N}{N^2} \Rightarrow \frac{d}{2}\big(1-p_0\big)\tau_{\mathbf{a},D} \quad \text{when } d \geq 3.$$

Moreover, for $k \geq 1$, we have that the kth moments of $R_N/(N^2/\log N)$ when $d = 2$ and those of R_N/N^2 when $d \geq 3$ converge to the kth moments of their distributional limits, which satisfy a system of Poisson PDE via (1) and (2).

Theorem 1.2 (Multiple range)**.** *Let $p \geq 1$. We have the weak convergence, as $N \uparrow \infty$, that*

$$\frac{R_N^{(p)}}{N^2/\log^2 N} \Rightarrow 2\pi^2\tau_{\mathbf{a},D} \quad \text{when } d = 2, \quad \text{and}$$
$$\frac{R_N^{(p)}}{N^2} \Rightarrow \frac{d}{2}\big(1-p_0\big)^2(p_0)^{p-1}\tau_{\mathbf{a},D} \quad \text{when } d \geq 3.$$

Moreover, for $k \geq 1$, we have that the kth moments of $R_N^{(p)}/(N^2/\log^2 N)$ when $d = 2$ and those of $R_N^{(p)}/N^2$ when $d \geq 3$ converge to the kth moments of their distributional limits, which satisfy a system of Poisson PDE via (1) and (3).

Let $\mathbb{P}_{N,\mathbf{b}}$ and $\mathbb{E}_{N,\mathbf{b}}$ be the probability measure and expectation governing the random walk path $X_\cdot^{(N)}$ where $X_0^{(N)} = \mathbf{b}$. Let also $\mathbb{P}_{\mathbf{b}}$ and $\mathbb{E}_{\mathbf{b}}$ be the process measure and expectation with respect to d-dimensional standard Brownian motion starting from $\mathbf{b}$.

Define, for $k \geq 1$, that

$$u^{(k)}(\mathbf{a}) = \mathbb{E}_{\mathbf{a}}\Big[\big(\tau_{\mathbf{a},D}\big)^k\Big].$$

Such moments are known to be 'polyharmonic', that is they satisfy a following system of PDE [10]: Let $k \geq 1$. Then, $u^{(k)}(\mathbf{a}) = 0$ for $\mathbf{a} \notin D$ and, for $\mathbf{a} \in D$, we have

$$\Delta u^{(1)} = -2$$
$$\Delta u^{(k+1)} = -2(k+1)u^{(k)}. \tag{1}$$

In particular, the limits

$$\lim_{N\uparrow\infty} \mathbb{E}_{N,\mathbf{a}_N}\Big[\Big(R_N/(N^2/\log N)\Big)^k\Big] = \pi^k u^{(k)}(\mathbf{a}) \quad \text{when } d = 2,$$
$$\lim_{N\uparrow\infty} \mathbb{E}_{N,\mathbf{a}_N}\Big[\Big(R_N/N^2\Big)^k\Big] = \Big(\frac{d}{2}(1-p_0)\Big)^k u^{(k)}(\mathbf{a}) \quad \text{when } d \geq 3. \tag{2}$$

Also, for $p \geq 1$,

$$\lim_{N\uparrow\infty} \mathbb{E}_{N,\mathbf{a}_N}\left[\left(R_N^{(p)}/(N^2/\log N)\right)^k\right] = (2\pi^2)^k u^{(k)}(\mathbf{a}) \ \text{ when } d = 2,$$

$$\lim_{N\uparrow\infty} \mathbb{E}_{N,\mathbf{a}_N}\left[\left(R_N^{(p)}/N^2\right)^k\right] = \left(\frac{d}{2}(1-p_0)^2(p_0)^{p-1}\right)^k u^{(k)}(\mathbf{a}) \ \text{ when } d \geq 3. \tag{3}$$

Remark 1.3 (Generalizations)**.** We comment on some generalizations.

Mean-zero walks. The proofs for Theorems 1.1 and 1.2 carry over straightforwardly to finite-range mean-zero random walks, with covariance matrix Γ, but now using also $\{\cos(n\pi\cdot) : n \geq 0\}$ in addition to $\{\sin(n\pi\cdot) : n \geq 0\}$ as bases functions in the Fourier decompositions. The statements would be similar to Theorems 1.1 and 1.2, but the term $d\tau_{\mathbf{a},D}$ would be replaced by the exit time from D of a Brownian motion with covariance matrix Γ starting from $\mathbf{a}$.

Biased walks. One might consider biased finite-range random walks with (vector) mean $\mathbf{m} \neq 0$. Adapting the proofs given here, one can show, in $d \geq 2$, that $R_N/N \to (1-p_0)c(\mathbf{a}, D)$ and $R_N^{(p)}/N \to (1-p_0)^2(p_0)^p c(\mathbf{a}, D)$ in probability, where $c(\mathbf{a}, D) = \inf\{c > 0 \,|\, \mathbf{a} + c\,\mathbf{m} \in D^c\}$.

Remark 1.4 (Dimension $d = 1$)**.** To compare, we specify the limit for the scaled range up to the time of exit from $(0, N)$ in $d = 1$ among other local time results in [4]: Namely $R_N/N \Rightarrow \zeta_{\mathbf{a}}$ where $\zeta_{\mathbf{a}} \neq \tau_{\mathbf{a},(0,1)}$ and has density $f(x) = (\mathbf{a} \wedge (1-\mathbf{a}))/x^2$ for $\mathbf{a} \wedge (1-\mathbf{a}) < x < \mathbf{a} \vee (1-\mathbf{a})$, $f(x) = 1/x^2$ for $\mathbf{a} \vee (1-\mathbf{a}) \leq x \leq 1$ and $f(x) = 0$ for x otherwise.

In the next Section 2, the proofs of Theorems 1.1 and 1.2 are given, with the aid of estimates in Sections 3 and 4.

2. Proof of Theorems 1.1 and 1.2

We now detail three ingredients, two of which are known, used in the proof of Theorems 1.1 and 1.2.

(A) Asymptotics of $\mathcal{R}_n$ and $\mathcal{R}_n^{(p)}$. It is known when $d = 2$ that $R_n/(n/\log n) \to \pi$ a.s. [7]. Whereas when $d \geq 3$, since the random walk is transient, $R_n/n \to 1 - p_0$ a.s. where p_0 is the probability of return to the starting point [7], [16, p. 38-40].

Also, for $p \geq 1$, when $d = 2$, it is known that $\mathcal{R}_n^{(p)}/(n/\log^2(n)) \to \pi^2$ a.s. [8]. But, when $d \geq 3$, $\mathcal{R}_n^{(p)}/n \to (1-p_0)^2(p_0)^{p-1}$ a.s. [15].

We note all of these results do not depend on the starting point value as the random walk dynamics on $\mathbb{Z}^d$ is translation-invariant.

(B) Input from a functional CLT. When $\mathbf{a}_N/N \to \mathbf{a} \in D$, we have by the functional central limit theorem that the random walk paths $\{\frac{\sqrt{d}}{N} X^{(N)}_{[N^2 s]} : s \geq 0; X_0^{(N)} = \mathbf{a}_N\}$ converge weakly say in the uniform topology to Brownian motion $\{B_s : s \geq 0; B_0 = \mathbf{a}\}$ (cf. Sections 16, 18 [6]).

Since D is an open, bounded and Lipschitz (or Jordan in $d = 2$) domain, the time $\tau_{\mathbf{a},D} < \infty$ is a continuous function with respect to the uniform topology on the space of Brownian trajectories a.s. Indeed, let ω be a Brownian trajectory starting from $\mathbf{a}$, and $\{\omega^n\}$ be a sequence of continuous paths converging to it uniformly on compact time intervals. One cannot have the limit $u = \lim \tau_{\mathbf{a},D}(\omega^n) < \tau_{\mathbf{a},D}(\omega) = v$, since then, from the uniform convergence, $u \geq \tau_{\mathbf{a},D}(\omega) = v$. But, one cannot have $u > v$ either, since from the uniform convergence, $\omega(r) \in \bar{D}$ for $v \leq r \leq u$, a contradiction given that ω must also visit $\mathbb{R}^d \setminus \bar{D}$ in this time interval as the Lipschitz (or Jordan in $d = 2$) boundary ∂D satisfies (after a conformal transformation in $d = 2$) a uniform cone condition (cf. [2, Ch. 4], [14, Ch. 4]).

Then, by the continuous mapping theorem, we have that

$$\frac{\tau_N}{N^2} \Rightarrow d\tau_{\mathbf{a},D}. \tag{4}$$

In particular, we have the convergence in probability as an immediate consequence,

$$\frac{\log \tau_N}{\log N} \xrightarrow{P} 2. \tag{5}$$

(C) Moment estimates. We show in Sections 3 and 4, with the aid of a 'conductance' estimate, the following limits. When $\mathbf{a}_N/N \to \mathbf{a} \in D$, we claim that

$$\limsup_{N\uparrow\infty} \mathbb{E}_{N,\mathbf{a}_N}\Big[\Big(\frac{R_N}{N^2/\log N}\Big)^k\Big] < \infty \quad \text{when } d = 2 \tag{6}$$

$$\limsup_{N\uparrow\infty} \mathbb{E}_{N,\mathbf{a}_N}\Big[\Big(\frac{R_N}{N^2}\Big)^k\Big] < \infty \quad \text{when } d \geq 3. \tag{7}$$

Moreover, we claim that

$$\limsup_{N\uparrow\infty} \mathbb{E}_{N,\mathbf{a}_N}\Big[\Big(\frac{R_N^{(p)}}{N^2/\log^2 N}\Big)^k\Big] < \infty \quad \text{when } d = 2 \tag{8}$$

$$\limsup_{N\uparrow\infty} \mathbb{E}_{N,\mathbf{a}_N}\Big[\Big(\frac{R_N^{(p)}}{N^2}\Big)^k\Big] < \infty \quad \text{when } d \geq 3. \tag{9}$$

Although not needed for Theorems 1.1 and 1.2, corresponding positive lower bounds can also be shown by arguments similar to those given for the upper bounds:

$$\liminf_{N\uparrow\infty} \mathbb{E}_{N,\mathbf{a}_N}[(R_N/w_N)^k] > 0 \ \text{ and } \ \liminf_{N\uparrow\infty} \mathbb{E}_{N,\mathbf{a}_N}[(R_N^{(p)}/v_N)^k] > 0, \tag{10}$$

where $w_N = N^2/\log(N)$ and $v_N = N^2/\log^2(N)$ in $d = 2$, and $w_N = v_N = N^2$ in $d \geq 3$. As a note to the interested reader, by Jensen's inequality, it suffices to show these claims (10) for $k = 1$.

Proof of Theorems 1.1 and 1.2. In terms of the 'ingredients', we now combine them in the following way. Suppose $d = 2$ and write

$$\frac{R_N}{N^2/\log N} = \frac{\mathcal{R}_{\tau_N}}{\tau_N/\log \tau_N}\frac{\tau_N}{N^2}\frac{\log N}{\log \tau_N}.$$

since $\tau_N \uparrow \infty$ a.s., by ingredient **A**, we have that $\mathcal{R}_{\tau_N}\log\tau_N/\tau_N \to \pi$ converges in probability. On the other hand, from ingredient **B**, we have that $(\tau_N/N^2)(\log N/\log \tau_N) \Rightarrow \tau_{\mathbf{a},D}$. Hence, $R_N/(N^2/\log N) \Rightarrow \pi\tau_{\mathbf{a},D}$ as desired.

The weak convergence argument for $R_N/N^2 \Rightarrow (d/2)(1-p_0)\tau_{\mathbf{a},D}$ when $d \geq 3$ is similar.

Then, with respect to Theorem 1.1, convergence of the moments follows immediately from the weak convergence and ingredient **C**.

Finally, we note that the argument for Theorem 1.2 follows the same steps. □

3. Moment bounds: Proofs of (6) and (7)

We will prove only (6), as the argument for (7) is easier, and in particular can be done in a similar way. Fix $d = 2$ for the remainder of the section.

The bound (6) will hold if we establish

$$B := \limsup_{N\uparrow\infty} \sup_{\mathbf{b}\in D_N} \frac{\log N}{N^2}\mathbb{E}_{N,\mathbf{b}}\big[R_N\big] < \infty \tag{11}$$

and, for $k \geq 2$, the factorial moment

$$\limsup_{N\uparrow\infty} \sup_{\mathbf{b}\in D_N} \Big(\frac{\log(N)}{N^2}\Big)^k \mathbb{E}_{N,\mathbf{b}}\Big[R_N(R_N-1)\cdots(R_N-(k-1))\Big] \leq B^k < \infty. \tag{12}$$

Preliminaries. Consider now the random walk hitting probability $P_{\mathbf{b}}^{(N)} : \mathbb{Z}^2 \to [0,1]$, for $\mathbf{b} \in D_N$, defined by

$$P_{\mathbf{b}}^{(N)}(\mathbf{x}) = \mathbb{P}_{N,\mathbf{b}}(T_{N,\mathbf{x}} < \tau_N) \tag{13}$$

where $T_{N,\mathbf{x}}$ is the random walk hitting time of $\mathbf{x} \in \mathbb{Z}^2$.

We now make a reduction step: Since $D \subset (-\frac{A}{2}, \frac{A}{2})^d$, for an integer $A < \infty$, we may bound $\mathbb{P}_{\mathbf{b}}^{(N)}(\mathbf{x}) \leq \mathbb{P}_{N,\mathbf{b}}(T_{N,\mathbf{x}} < \tau'_N)$ where τ'_N is the exit time from $NA(-\frac{1}{2}, \frac{1}{2})^2$. Then, $R_N \leq \mathcal{R}_{\tau'_{NA}}$ where τ'_{NA} is the time to exit from a cube of size 1, scaled by NA. Hence, it will be enough to show the bounds (6), (7), via translation-invariance, with respect to the cube $(0,1)^2$. Accordingly, we fix for the remainder of the section that $D = (0,1)^2$.

We now derive an explicit expression for $P_{\mathbf{b}}^{(N)}$. Note $P_{\mathbf{b}}^{(N)}(\mathbf{b}) = 1$ and, for $\mathbf{x} \notin D_N$, that $P_{\mathbf{b}}^{(N)}(\mathbf{x}) = 0$. On the other hand, if $\mathbf{x} \in D_N$ and $\mathbf{b} \neq \mathbf{x}$, then first step analysis yields

$$P_{\mathbf{b}}^{(N)}(\mathbf{x}) = \frac{1}{4}\left(P_{\mathbf{b}+\mathbf{e}_1}^{(N)}(\mathbf{x}) + P_{\mathbf{b}-\mathbf{e}_1}^{(N)}(\mathbf{x}) + P_{\mathbf{b}+\mathbf{e}_2}^{(N)}(\mathbf{x}) + P_{\mathbf{b}-\mathbf{e}_2}^{(N)}(\mathbf{x})\right). \tag{14}$$

Define a notion of 'conductance' $g_N : \mathbb{Z}^2 \to [0,1]$ by

$$\begin{aligned} g_N(\mathbf{x}) &= 1 - \frac{1}{4}\left(P_{\mathbf{x}+\mathbf{e}_1}^{(N)}(\mathbf{x}) + P_{\mathbf{x}-\mathbf{e}_1}^{(N)}(\mathbf{x}) + P_{\mathbf{x}+\mathbf{e}_2}^{(N)}(\mathbf{x}) + P_{\mathbf{x}-\mathbf{e}_2}^{(N)}(\mathbf{x})\right), \\ &= \mathbb{P}_{\mathbf{x}}\big(\tau_N < T_{N,\mathbf{x}}\big), \end{aligned} \tag{15}$$

so that for $\mathbf{x} \in D_N$, we have

$$P_{\mathbf{b}}^{(N)}(\mathbf{x}) - \frac{1}{4}\left(P_{\mathbf{b}+\mathbf{e}_1}^{(N)}(\mathbf{x}) + P_{\mathbf{b}-\mathbf{e}_1}^{(N)}(\mathbf{x}) + P_{\mathbf{b}+\mathbf{e}_2}^{(N)}(\mathbf{x}) + P_{\mathbf{b}-\mathbf{e}_2}^{(N)}(\mathbf{x})\right) = g_N(\mathbf{x})\, 1_{\mathbf{b}}(\mathbf{x}). \tag{16}$$

Write $\mathbf{b} = (a,b)$ and consider now the discrete Fourier transforms of each term in (16):

$$P_{\mathbf{b}}^{(N)}(\mathbf{x}) = \sum_{m=1}^{N-1}\sum_{n=1}^{N-1} a_{mn} \sin\left(\frac{n\pi a}{N}\right) \sin\left(\frac{m\pi b}{N}\right),$$

$$\begin{aligned} P_{\mathbf{b}\pm\mathbf{e}_1}^{(N)}(\mathbf{x}) &= \sum_{m=1}^{N-1}\sum_{n=1}^{N-1} a_{mn} \sin\left(\frac{n\pi(a\pm 1)}{N}\right) \sin\left(\frac{m\pi b}{N}\right), \\ P_{\mathbf{b}\pm\mathbf{e}_2}^{(N)}(\mathbf{x}) &= \sum_{m=1}^{N-1}\sum_{n=1}^{N-1} a_{mn} \sin\left(\frac{n\pi a}{N}\right) \sin\left(\frac{m\pi(b\pm 1)}{N}\right), \\ g_N(\mathbf{x}) 1_{\mathbf{b}}(\mathbf{x}) &= \sum_{m=1}^{N-1}\sum_{n=1}^{N-1} b_{mn} \sin\left(\frac{n\pi a}{N}\right) \sin\left(\frac{m\pi b}{N}\right). \end{aligned}$$

Through straightforward trigonometric manipulations, (16) is re-expressed as

$$\sum_{m=1}^{N-1}\sum_{n=1}^{N-1} a_{mn} \sin\left(\frac{n\pi a}{N}\right) \sin\left(\frac{m\pi b}{N}\right)\left(1 - \frac{1}{2}\left[\cos\left(\frac{n\pi}{N}\right) + \cos\left(\frac{m\pi}{N}\right)\right]\right) = g_N(\mathbf{x})\, 1_{\mathbf{b}}(\mathbf{x}). \tag{17}$$

We now find formulas for a_{mn} and b_{mn}. Recall the orthogonality relation, for $n, n' \in \mathbb{N}$, that

$$\sum_{\ell=1}^{N-1} \sin\left(\frac{n\pi\ell}{N}\right) \sin\left(\frac{n'\pi\ell}{N}\right) = \frac{N 1(n = n')}{2}.$$

Then,

$$\sum_{r=1}^{N-1}\sum_{\ell=1}^{N-1} g_N(\mathbf{x})\, 1_{\mathbf{b}}(\mathbf{x})\, \sin\left(\frac{n'\pi\ell}{N}\right) \sin\left(\frac{m'\pi r}{N}\right) = \frac{N^2 b_{m'n'}}{4}.$$

Hence, writing $\mathbf{x} = (x_1, x_2)$, we have

$$b_{mn} = \frac{4 g_N(x,y) \sin\left(\frac{n\pi x_1}{N}\right) \sin\left(\frac{m\pi x_2}{N}\right)}{N^2}.$$

Moreover, by equating the coefficients $a_{mn}[(1-(1/2)[\cos(n\pi/N)+\cos(m\pi/N)])] = b_{mn}$ with respect to (17), we have

$$a_{mn} = \frac{4g_N(\mathbf{x})\sin\left(\frac{n\pi x_1}{N}\right)\sin\left(\frac{m\pi x_2}{N}\right)}{N^2\left(1-\frac{1}{2}\left[\cos\left(\frac{n\pi}{N}\right)+\cos\left(\frac{m\pi}{N}\right)\right]\right)}.$$

We now obtain explicit formulas for $g_N(\mathbf{x})$ and $P^{(N)}_{\mathbf{b}}$. The boundary condition $P^{(N)}_{\mathbf{x}}(\mathbf{x}) = 1$ yields

$$P^{(N)}_{\mathbf{x}}(\mathbf{x}) = \sum_{m=1}^{N-1}\sum_{n=1}^{N-1} a_{mn}\sin\left(\frac{n\pi x_1}{N}\right)\sin\left(\frac{m\pi x_2}{N}\right) = 1,$$

from which

$$g_N(\mathbf{x}) = \left[\sum_{m=1}^{N-1}\sum_{n=1}^{N-1}\frac{4\sin^2\left(\frac{n\pi x_1}{N}\right)\sin^2\left(\frac{m\pi x_2}{N}\right)}{N^2\left(1-\frac{1}{2}\left[\cos\left(\frac{n\pi}{N}\right)+\cos\left(\frac{m\pi}{N}\right)\right]\right)}\right]^{-1}$$

and

$$P^{(N)}_{\mathbf{b}}(\mathbf{x}) = \sum_{m=1}^{N-1}\sum_{n=1}^{N-1}\frac{4g_N(\mathbf{x})\sin\left(\frac{n\pi x_1}{N}\right)\sin\left(\frac{m\pi x_2}{N}\right)\sin\left(\frac{n\pi a}{N}\right)\sin\left(\frac{m\pi b}{N}\right)}{N^2\left(1-\frac{1}{2}\left[\cos\left(\frac{n\pi}{N}\right)+\cos\left(\frac{m\pi}{N}\right)\right]\right)}. \tag{18}$$

We now claim the following upper bound, proved at the end of the subsection,

$$\limsup_{N\uparrow\infty}\ \sup_{N/\log^2(N)\le x_1,x_2\le N-N/\log^2(N)} \log(N)g_N(\mathbf{x}) < \infty. \tag{19}$$

We are now in position to show (11) and (12).

Proof of (11)**.** We may write the expected range $\mathbb{E}_{N,\mathbf{b}}[R_N]$ as

$$\mathbb{E}_{N,\mathbf{b}}[R_N] = \mathbb{E}_{N,\mathbf{b}}\left[\sum_{\mathbf{x}\in D_N} 1(T_{N,\mathbf{x}} < \tau_N)\right] = \sum_{\mathbf{x}\in D_N} P^{(N)}_{\mathbf{b}}(\mathbf{x}).$$

Let

$$D^1_N = \{\mathbf{x}\in D_N : N/\log^2(N) \le x_1, x_2 \le N - N/\log^2(N)\}. \tag{20}$$

Then,

$$\mathbb{E}_{N,\mathbf{b}}[R_N] = \sum_{\mathbf{x}\in D^1_N} P^{(N)}_{\mathbf{b}}(\mathbf{x}) + \sum_{\mathbf{x}\in D_N\cap(D^1_N)^c} P^{(N)}_{\mathbf{b}}(\mathbf{x}) = J_1 + J_2.$$

The second sum J_2, bounding $P^{(N)}_{\mathbf{b}}(\mathbf{x}) \le 1$, is of order $O(N^2/\log^2(N))$.

We now show an $O(N^2/\log(N))$ bound for the first sum J_1 to finish. Note the limit

$$\lim_{N\to\infty}\frac{1}{N}\sum_{\ell=1}^{N}\sin(r\pi\ell/N) = \int_0^1 \sin(r\pi u)du = \frac{1(r \text{ odd})}{\pi r}.$$

With the formula (18) and bound (19) in hand, uniformly over $\mathbf{b}$, we conclude

$$\begin{aligned} J_1 &\le O(N^2/\log N)\sum_{\mathbf{x}\in D^1_N}\sum_{1\le n,m\le N}\frac{\sin(n\pi x_1/N)\sin(m\pi x_1/N)}{N^2[n^2+m^2]} \\ &\le O(N^2/\log N)\sum_{1\le n,m\le N}\frac{1}{nm(n^2+m^2)} = O(N^2/\log N), \end{aligned}$$

as desired. □

Proof of (12)**.** For $1\le\ell\le k-1$, define and bound

$$P^{(N)}_{\mathbf{x}}(\mathbf{y};\{\mathbf{z}_\ell,\dots,\mathbf{z}_k\}) = \mathbb{P}_{\mathbf{x}}(T_{N,\mathbf{y}} < \min\{T_{N,\mathbf{z}_\ell},\dots,T_{N,\mathbf{z}_k},\tau_N\}) \le P^{(N)}_{\mathbf{x}}(\mathbf{y}).$$

Denote by $\mathbb{S}_k$ the set of permutations of $\{1, 2, \ldots, k\}$. The factorial moment of R_N of order k is writtten as

$$\begin{aligned}
&\mathbb{E}_{N,\mathbf{b}}\Big[R_N\big(R_N-1\big)\cdots\big(R_N-(k-1)\big)\Big] \\
&\quad = \mathbb{E}_{N,\mathbf{b}}\Big[\sum_{\substack{\mathbf{z}_1,\ldots,\mathbf{z}_k\in D_N\\ \text{distinct}}}\prod_{\ell=1}^k 1(T_{\mathbf{z}_\ell}<\tau_N)\Big] \\
&\quad = \sum_{\substack{\mathbf{z}_1,\ldots,\mathbf{z}_k\in D_N\\ \text{distinct}}}\sum_{\pi\in\mathbb{S}_k}\mathbb{P}_{\mathbf{b}}(\mathbf{z}_{\pi_1};\{\mathbf{z}_{\pi_2},\ldots,\mathbf{z}_{\pi_k}\})\mathbb{P}_{\mathbf{z}_{\pi_1}}(\mathbf{z}_{\pi_2};\{\mathbf{z}_{\pi_3},\ldots,\mathbf{z}_{\pi_k}\})\cdots\mathbb{P}_{\mathbf{z}_{\pi_{k-1}}}(\mathbf{z}_{\pi_k}).
\end{aligned}$$

We now observe, for each $\pi\in\mathbb{S}_k$, by the proven (11), that

$$\begin{aligned}
&\sum_{\substack{\mathbf{z}_1,\ldots,\mathbf{z}_k\in D_N\\ \text{distinct}}}\mathbb{P}_{\mathbf{b}}(\mathbf{z}_{\pi_1};\{\mathbf{z}_{\pi_2},\ldots,\mathbf{z}_{\pi_k}\})\mathbb{P}_{\mathbf{z}_{\pi_1}}(\mathbf{z}_{\pi_2};\{\mathbf{z}_{\pi_3},\ldots,\mathbf{z}_{\pi_k}\})\cdots\mathbb{P}_{\mathbf{z}_{\pi_{k-1}}}(\mathbf{z}_{\pi_k}) \\
&\qquad\le \sum_{\substack{\mathbf{z}_1,\ldots,\mathbf{z}_k\in D_N\\ \text{distinct}}}\mathbb{P}_{\mathbf{b}}(\mathbf{z}_{\pi_1})\mathbb{P}_{\mathbf{z}_{\pi_1}}(\mathbf{z}_{\pi_2})\cdots\mathbb{P}_{\mathbf{z}_{\pi_{k-1}}}(\mathbf{z}_{\pi_k}) \\
&\qquad= \mathbb{E}_{N,\mathbf{b}}[R_N]\prod_{\ell=2}^k\mathbb{E}_{N,\mathbf{z}_{\pi_{\ell-1}}}[R_N] = O\big((N^2/\log(N))^k\big).
\end{aligned}$$

Hence, as a consequence, (12) follows immediately. ☐

Proof of (19)**.** Since $1-\cos(n\pi/N) = 2\sin^2(n\pi/(2N))$, $2\sin^2(\ell\pi/(2N)) \le 2\ell^2\pi^2/(4N^2)$, we have

$$g_N^{-1}(\mathbf{x}) \ge \frac{2}{\pi^2}\sum_{n=1}^N\sum_{m=1}^N\frac{\sin^2(n\pi x_1/N)\sin^2(m\pi x_2/N)}{n^2+m^2}.$$

Note, when $\mathbf{x}$ is the midpoint, $\mathbf{x} = (\lfloor N/2\rfloor, \lfloor N/2\rfloor)$,

$$g_N^{-1}(\mathbf{x}) \ \ge\ \frac{2}{\pi^2}\sum_{\substack{1\le n,m\le N\\ \text{odd}}}\frac{1}{n^2+m^2}\ \ge (1/\pi^2)\log(N).$$

When $\mathbf{x}$ is not the midpoint, let $d_N = d_N(\mathbf{x})$ be its distance to the boundary of D_N. Let $B_N = B_N(\mathbf{x}) \subset D_N$ be the cube with width d_N and center $\mathbf{x}$. Let $\zeta_N = \zeta_N(\mathbf{x})$ be the time for the random walk to exit B_N. Since $\zeta_N \le \tau_N$, by (15), we have that $g_N(\mathbf{x}) \le \mathbb{P}_{\mathbf{x}}(\zeta_N < T_{N,\mathbf{x}}) := \widetilde{g}_N(\mathbf{x})$.

One can derive, a formula and lower bound for $\widetilde{g}_N(\mathbf{x})$, as for $g_N(\mathbf{x})$ when $\mathbf{x}$ is the midpoint above, using the scale d_N and domain B_N instead of N and D_N:

$$\begin{aligned}
\widetilde{g}_N^{-1}(\mathbf{x}) &= \sum_{n=1}^{d_N}\sum_{m=1}^{d_N}\frac{4\sin^2\left(\frac{n\pi x_1}{d_N}\right)\sin^2\left(\frac{n\pi x_2}{d_N}\right)}{N^2\big(1-\frac{1}{2}\big[\cos\left(\frac{n\pi}{d_N}\right)+\cos\left(\frac{m\pi}{d_N}\right)\big]\big)} \\
&\ge (1/\pi^2)\log(d_N).
\end{aligned}$$

Hence, when $N/\log^2(N) \le x_1, x_2 \le N - N/\log^2(N)$, we have that

$$g_N^{-1}(\mathbf{x}) \ge \widetilde{g}_N^{-1}(\mathbf{x}) \ge (1/2\pi^2)\log(N) \tag{21}$$

for all large N, as desired. ☐

4. Moment bounds: Proofs of (8) and (9)

For $d \geq 3$, the bound in (9) follows from (7) since $R_N^{(p)} \leq R_N$ a.s. So we only need to show (8) when $d = 2$. We write

$$\mathbb{E}_{N,\mathbf{b}}[R_N^{(p)}] \;=\; \sum_{\mathbf{x}\in D_N} \mathbb{E}_{N,\mathbf{b}}\Big[1\big(T_{N,\mathbf{x}}^{(p)} < \tau_N < T_{N,\mathbf{x}}^{(p+1)}\big)\Big] \;=\; \sum_{\mathbf{x}\in D_N} Q_{\mathbf{b}}^{(N)}(\mathbf{x})$$

where $Q_{\mathbf{b}}^{(N)} : \mathbb{Z}^d \to [0,1]$ is given by

$$\begin{aligned} Q_{\mathbf{b}}^{(N)}(\mathbf{x}) &= \mathbb{P}_{N,\mathbf{b}}(T_{N,\mathbf{x}}^{(p)} < \tau_N < T_{N,\mathbf{x}}^{(p+1)}) \\ &\leq \mathbb{P}_{N,\mathbf{b}}(T_{N,\mathbf{x}}^{(p)} < \tau_N)\mathbb{P}_{N,\mathbf{x}}(\tau_N < T_{N,\mathbf{x}}). \end{aligned}$$

As $P_{\mathbf{b}}^{(N)}(T_{N,\mathbf{x}}^{(p)} < \tau_N) \leq \mathbb{P}_{N,\mathbf{b}}(T_{N,\mathbf{x}} < \tau_N) = P_{\mathbf{b}}^{(N)}(\mathbf{x})$ and $g_N(\mathbf{x}) = \mathbb{P}_{N,\mathbf{x}}(\tau_N < T_{N,\mathbf{x}})$ (cf. (15)), we have

$$Q_{\mathbf{b}}^{(N)}(\mathbf{x}) \leq g_N(\mathbf{x})\mathbb{P}_{\mathbf{b}}^{(N)}(\mathbf{x}).$$

Recall $\widetilde{g}_N$ from the proof of (19). Note that $g_N(\mathbf{x}) \leq \widetilde{g}_N(\mathbf{x})$ and that $\widetilde{g}_N(\mathbf{x}) = O(\log^{-1}(N))$ when d_N, the distance between $\mathbf{x}$ and the boundary of D_N, is greater than $N/\log^2(N)$ say.

Consider now the set $D_N^2 \subset D_N$ of points $\mathbf{x}$ away by at least $N/\log^2(N)$ from the boundary of D_N. From our assumptions on D, since ∂D has finite perimeter, the area of the region within $N\log^{-2}(N)$ of ∂D is of order $O(\mathrm{Per}(\partial D)\, N\log^{-2}(N))$, and so the number of points in D_N within distance $N\log^{-2}(N)$ is of order $O(N^2/\log^2(N))$.

Recall, by the proven (6) that $\mathbb{E}_{N,\mathbf{b}}[R_N] = O(N^2/\log(N))$ uniformly over $\mathbf{b} \in D_N$. We then obtain, uniformly over $\mathbf{b} \in D_N$, that

$$\begin{aligned} \mathbb{E}_{N,\mathbf{b}}[R_N^{(p)}] &\leq O(N^2/\log^2(N)) + \sum_{\mathbf{x}\in D_N^2} \widetilde{g}_N(\mathbf{x}) P_{\mathbf{b}}^{(N)}(\mathbf{x}) \\ &\leq O(N^2/\log^2(N)) + O(\log^{-1}(N))\mathbb{E}_{N,\mathbf{b}}[R_N] \;= O(N^2/\log^2(N)). \end{aligned}$$

This gives (8) with respect to $k = 1$.

We now turn to estimating the factorial moment of order $k \geq 2$. We concentrate on the case $k = 2$ which encapsulates the main ideas, and then comment on the case $k \geq 3$.

Consider that

$$\begin{aligned} &\mathbb{P}_{N,b}(T_{N,\mathbf{x}}^{(p)}, T_{N,\mathbf{y}}^{(p)} < \tau_N < T_{N,\mathbf{x}}^{(p+1)}, T_{N,\mathbf{y}}^{(p+1)}) \\ &\quad = \mathbb{P}_{N,b}(T_{N,\mathbf{x}}^{(p)} < T_{N,\mathbf{y}}^{(p)} < \tau_N < T_{N,\mathbf{x}}^{(p+1)}, T_{N,\mathbf{y}}^{(p+1)}) + \mathbb{P}_{N,b}(T_{N,\mathbf{y}}^{(p)} < T_{N,\mathbf{x}}^{(p)} < \tau_N < T_{N,\mathbf{x}}^{(p+1)}, T_{N,\mathbf{y}}^{(p+1)}). \end{aligned} \tag{22}$$

We bound now the first term on the right-hand side of (22). Let again B_N be a cube centered at x with width $d_N(\mathbf{x}) = O(N/\log^2(N))$ and ζ_N be the exit time from B_N. In decomposing the path specified in the first term, we argue, for $y \notin B_N$, that

$$\begin{aligned} &\mathbb{P}_{N,b}(T_{N,\mathbf{x}}^{(p)} < T_{N,\mathbf{y}}^{(p)} < \tau_N < T_{N,\mathbf{x}}^{(p+1)}, T_{N,\mathbf{y}}^{(p+1)}) \\ &\qquad \leq p \times \mathbb{P}_{N,\mathbf{b}}(T_{N,\mathbf{x}} < \tau_N)\mathbb{E}_{N,\mathbf{x}}[1(\zeta_N < T_{N,\mathbf{x}})\mathbb{P}_{N,X_{\zeta_N}^{(N)}}(T_{N,\mathbf{y}} < \tau_N)]\mathbb{P}_{N,\mathbf{y}}(\tau_N < T_{N,\mathbf{y}}). \end{aligned} \tag{23}$$

We obtain a similar expression bounding the second term in (22) by interchanging $\mathbf{x}$ and $\mathbf{y}$. Indeed, the left-hand side of (23) is the probability of the set of paths that start at b, hit x exactly p times before the p-th visit to y after which the boundary ∂D_N is hit before coming back to x or y. The set of these paths is a subset of paths, starting from b, hitting x (exactly p times and hitting y at most $\ell(\mathbf{y}) \leq p-1$ times), which then exit a box B_N centered at x (without hitting x again) to hit y (exactly $p - \ell(\mathbf{y})$ times) before hitting ∂D_N. The probability of the set of such paths is bounded by the right-hand side expression.

By (21), $\mathbb{P}_{N,\mathbf{x}}(\zeta_N < T_{N,\mathbf{x}}) = O(\log^{-1}(d_N(\mathbf{x})))$. Summing (23), and its analog with $\mathbf{x}$ and $\mathbf{y}$ interchanged, over $\mathbf{x}$ and $\mathbf{y}$, separated by at least $N/\log^2(N)$ and also when both $\mathbf{x}, \mathbf{y}$ are away from the boundary of D_N by $N\log^2(N)$, noting the proved estimate (11), we obtain uniformly over $\mathbf{b} \in D_N$ the desired bound on the factorial moment

$$\mathbb{E}_{N,\mathbf{b}}[R_N^{(p)}(R_N^{(p)}-1)] = O(N^2/\log^2(N)) + O(\sup_{\mathbf{z}\in D_N} \mathbb{E}_{N,\mathbf{z}}[R_N]^2/\log^2(N)) = O(N^4/\log^4(N)).$$

When $k \geq 3$, analogously, we may bound $k!$ terms such as

$$\mathbb{P}_{N,\mathbf{b}}(T^{(p)}_{N,\mathbf{z}_1} < T^{(p)}_{N,\mathbf{z}_2} < \cdots < T^{(p)}_{N,\mathbf{z}_k} < \tau_N < T^{(p+1)}_{N,\mathbf{z}_1}, \ldots, T^{(p+1)}_{N,\mathbf{z}_k}). \tag{24}$$

Indeed, consider cubes of width $O(N/\log^2(N))$ centered around $\mathbf{z}_1, \ldots, \mathbf{z}_{k-1}$. Let $\zeta_{1,N}, \ldots, \zeta_{k-1,N}$ be the exit times from these cubes. Then, for $\mathbf{z}_1, \ldots, \mathbf{z}_k$ separated from each other and the boundary of D_N by $O(N/\log^2(N))$, we have (24) is bounded by

$$\begin{aligned}
p^{k-1} &\times \mathbb{P}_{N,\mathbf{b}}(T_{N,\mathbf{z}_1} < \tau_N)\mathbb{E}_{N,\mathbf{z}_1}[1(\zeta_{1,N} < T_{N,\mathbf{z}_1})\mathbb{P}_{N,X^{(N)}_{\zeta_{1,N}}}(T_{N,\mathbf{z}_2} < \tau_N)] \\
&\times \mathbb{E}_{N,\mathbf{z}_2}[1(\zeta_{2,N} < T_{N,\mathbf{z}_2})\mathbb{P}_{N,X^{(N)}_{\zeta_{2,N}}}(T_{N,\mathbf{z}_3} < \tau_N)] \cdots \\
&\cdots \times \mathbb{E}_{N,\mathbf{z}_{k-2}}[1(\zeta_{k-1,N} < T_{N,\mathbf{z}_{k-1}})\mathbb{P}_{N,X^{(N)}_{\zeta_{k-1,N}}}(T_{N,\mathbf{z}_k} < \tau_N)]\mathbb{P}_{N,\mathbf{z}_k}(\tau_N < T_{N,\mathbf{z}_k}).
\end{aligned}$$

Through the estimate (21) again, summing such an expression over $\mathbf{z}_1, \ldots, \mathbf{z}_k$ we can bound the factorial moment uniformly over $\mathbf{b} \in D_N$ as

$$\begin{aligned}
\mathbb{E}_{N,\mathbf{b}}[R_N^{(p)}(R_N^{(p)}-1)\cdots(R_N^{(p)}-(k-1))] &= O(N^k/\log^2(N)) + O((\sup_{\mathbf{z}\in D_N} \mathbb{E}_{N,\mathbf{z}}[R_N])^k/\log^k(N)) \\
&= O((N^2/\log^2(k))^k).
\end{aligned}$$

Hence, in this way, (8) follows. $\square$

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