# CONDENSATION, BOUNDARY CONDITIONS, AND EFFECTS OF SLOW SITES IN ZERO-RANGE SYSTEMS

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Abstract. We consider the space-time scaling limit of the particle mass in zero-range particle systems on a 1D discrete torus $\mathbb{Z}/N\mathbb{Z}$ with a finite number of defects. We focus on two classes of increasing jump rates $g$, when $g(n) \sim n^\alpha$, for $\alpha > 0$, and when $g$ is a bounded function. In such a model, a particle at a regular site $k$ jumps equally likely to a neighbor with rate $g(n)$, depending only on the number of particles $n$ at $k$. At a defect site $k_{j,N}$, however, the jump rate is slowed down to $\lambda_j^{-1}N^{-\beta_j}g(n)$ when $g(n) \sim n^\alpha$, and to $\lambda_j^{-1}g(n)$ when $g$ is bounded. Here, $N$ is a scaling parameter where the grid spacing is seen as $1/N$ and time is speeded up by $N^2$.

We will start from initial measures with $O(N)$ relative entropy with respect to an invariant measure. For rates $g(n) \sim n^\alpha$, we find that the hydrodynamic limit is written in terms of three types of PDE, when $\beta_j < \alpha$, $\beta_j = \alpha$, and $\beta_j > \alpha$, with associated Dirichlet boundary conditions at the macroscopic locations $x_j = \lim_{N\uparrow\infty} k_{j,N}/N$, reflecting interactions with evolving masses of atoms at the slow sites and condensation on them. However, when $g$ is bounded, at the macroscopic defect sites, we find the hydrodynamic density must be bounded by a threshold value, reflecting interactions with masses of atoms there.

## Contents



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## 1. Introduction

The purpose of this article is to understand the macroscopic boundary conditions which arise in hydrodynamic scaling limits for the space-time 'bulk' mass evolution of zero-range processes with a finite number of 'defects', and also related effects of 'condensation' at these defect locations. Such an aim more broadly fits into the study of how macroscopic boundary conditions emerge from inhomogeneous microscopic interactions.

In this view, there has been much interesting work on dimension $d = 1$ exclusion models where a site or bond is 'slowed' down. In [11], [12], [13], in computing the hydrodynamic limit in symmetric systems, different boundary conditions from Dirichlet to Neumann, and also Robin have been derived; see also [9]. There are however only a few works with respect to different interactions, in particular zero-range systems. Among these, [16] studies the hydrodynamic limit for a system of symmetric independent particles moving in a $d = 1$ (random) trap environment. In [18], with respect to a totally asymmetric zero-range process in $d = 1$, with bounded, increasing rate function $g(\cdot)$, effects of a slow site and slow particle are also found when starting from a 'flat' initial measure. In [4], with respect to a class of such totally asymmetric zero-range systems in $d = 1$, however with a nontrivial density of site disorders, hydrodynamics is shown with respect to an effective flux function, constant at supercritical densities.

We also mention that systems with 'reservoirs', typically in $d = 1$, have been studied–see [8] for a discussion of 'hydrostatics', and related references, with respect to exclusion models. In zero-range processes, 'static' reservoir effects are studied and dynamical conjectures are discussed in [14].

In this context, the general goal of our work is to consider the effect of a finite number of 'slow' sites in a class of symmetric zero-range systems in a $d = 1$ torus $\mathbb{Z}/N\mathbb{Z}$. In such a model, particles would 'condense' on a defect site if jump rates from it are 'slow' enough. One would expect that for the particle continuum mass, different boundary conditions would result depending on how 'slow' the defect is. Our main results describe corresponding hydrodynamic limits, in terms of a nonlinear parabolic PDE and evolving point masses, with specified boundary conditions at defects reflecting different types of condensation. The proof method, in the scheme of the 'entropy' method, however develops local 'replacements' which may be useful in other problems.

By specifying the locations of the 'slow' defects in the system, we fix the macroscopically separated points where 'condensation' can occur. There seems to be little work on the dynamical structure in such systems. This is in contrast to the well-developed study of 'condensation', which 'spontaneously' forms at a random location by introducing more particles in a zero-range system with a bounded rate function than is allowed to equilibriate. See [2] for a discussion of both mechanisms with respect to canonical and grand canonical measures invariant measures of a bounded rate zero-range process with a single defect.

We mention, among the recent literature on the type of 'condensation' which emerges at a random location, [3] consider, with respect to a thermodynamic limit in symmetric zero-range models on $L = L(N)$ sites in $d = 1$, the evolution of the random 'condensate' in a certain time-scale, as the density $N/L(N) \to \rho$. In [19], with respect to asymmetric dynamics in a set of $L$ fixed sites in $d = 1$, motion of the 'condensate' is described. In [5], a 'martingale problem' approach for the condensate dynamics is developed. In [22], some partial results on a hydrodynamic limit is given. See also references therein in these papers for a more complete history of the subject. For related notions of 'metastability', see books and surveys [6], [20], [23].

*Sketch of results.* To describe our results, we consider zero-range processes on the $d = 1$ torus $\mathbb{T}_N = \{0, 1, \ldots, N-1\}$ where 0 and $N$ are identified, that is $\mathbb{T}_N$, corresponding macroscopically to the unit torus $\mathbb{T}$. The jump rate functions $g : \mathbb{N}_0 \to [0, \infty)$ focused upon are in two forms (1) $g(n) \sim n^\alpha$ and $0 < \alpha \leq 1$, and when (2) $g$ is bounded (which includes the case $\alpha = 0$). In both settings, we will assume that $g$ is an increasing function, so that the process will allow 'attractive' couplings; see Section 3.2 for a discussion of its use.

In this process, at a regular site $k \in \mathbb{T}_N$, if there are $n$ particles there, one of them leaves with rate $g(n)$ to a neighbor, jumping either to $k-1$ or $k+1$, with equal probability. However, at a defect site $k$, the departure rate is altered in the following sense: Let $\lambda > 0$ and $\beta \in \mathbb{R}$. If there are $n$ particles at the defect $k$, one of them leaves at rate $\frac{1}{\lambda N^\beta} g(n)$. We will say the rate is 'slow' when $\beta > 0$, or $\beta = 0$ and $\lambda > 1$.

The zero-range system tracks the evolution of the unlabeled particles on $\mathbb{T}_N$. We denote by $\xi_t = \{\xi_t(k) : k \in \mathbb{T}_N\}$ the configuration of the process, where $\xi_t(k)$ is the number of particles at site $k$ at time $t \geq 0$. Given the symmetric transitions, it will be useful to define also the speeded-up process $\eta_t = \xi_{N^2 t}$. For this Markov system, there is a family of product (reversible) invariant measures $\mathscr{R}_c^N$, indexed by a 'density' parameter $0 \leq c < g_\infty = \lim_{n\to\infty} g(n)$; see Section 2.1.

For the system with rate $g(n) \sim n^\alpha$, we will start the process from measures $\mu^N$, associated to an initial macroscopic measure $\pi_0$ on $\mathbb{T} = [0, 1)$, with $O(N)$ relative entropy with respect to an invariant distribution $\mathscr{R}_{c_0}^N$, and stochastically bounded by another invariant distribution $\mathscr{R}_{c'}^N$. With respect to bounded rates $g$ (informally corresponding to $\alpha = 0$), $\mu^N$ satisfies a similar but slightly different criteria, since when $g_\infty < \infty$ there will be a finite effective critical density above which the invariant measure is not defined; see Condition 3.1. Here, the initial profile $\pi_0$ will be in form

$$\pi_0(dx) = \rho_0(x)dx + \sum_{j:\beta_j = \alpha} \mathfrak{m}_{0,j} \delta_{x_j}(dx).$$

Examples of suitable initial measures $\mu^N$ are given by local equilibrium product measures; see Section 3.3.

*Rates $g(n) \sim n^\alpha$.* To describe the main result in the setting $g(n) \sim n^\alpha$, it will be helpful to get a sense of the 'condensation' of particles, under an invariant measure $\mathscr{R}_c^N$. Typically at a slow site, the number of particles will be $O(N^{\beta/\alpha})$ when $\beta > \alpha$, order $O(N)$ when $\beta = \alpha$, and $o(N)$ when $\beta < \alpha$. So, with a finite number of slow sites, if one of them $k_{j,N}$ is such that $\beta_j > \alpha$, there will be a superlinear $O(N^{\beta/\alpha})$ number of particles in the system. Whereas, when all the $\beta_j \leq \alpha$, there will be $O(N)$ particles on $\mathbb{T}_N$. See Section 2.2 for precise statements.

Let $\mathfrak{D}_{s,N}$ be those slow sites $k_{j,N}$ where $\beta_j > \alpha$, and let $\mathfrak{D}_s$ be the corresponding set of continuum points $x_j \sim k_{j,N}/N$. Since, at these locations, the particle numbers are superlinear, denote the empirical measure with respect to the diffusively scaled system by

$$\pi_t^N = \frac{1}{N} \sum_{k \in \mathbb{T}_N \setminus \mathfrak{D}_{s,N}} \eta_t(k) \delta_{k/N}.$$

By a hydrodynamic limit, we mean, with respect to test functions $G \in C(\mathbb{T})$, that $\langle G, \pi_t^N \rangle$ converges in probability to $\langle G, \pi_t \rangle$, where $\pi_t$ is a measure-valued weak solution of a specified macroscopic evolution.

Our first result (Theorem 4.1) is that three different behaviors and boundary conditions, at the macroscopic defect sites $x_j$, arise in the hydrodynamic limit depending on when $\beta_j > \alpha$, $\beta_j = \alpha$ and $\beta_j < \alpha$. As might be suspected, when $\beta_j < \alpha$, the defect is not 'slow' enough to be seen in the continuum limit. However, defects where $\beta_j \geq \alpha$ on the other hand do register. To describe the macroscopic flow, segregate the unit torus $\mathbb{T}$ into intervals with endpoints $x_j$ corresponding to $\beta_j \geq \alpha$. In the interior of each interval, the hydrodynamic limit is described by a nonlinear heat equation

$$\partial_t \rho = \partial_{xx} \Phi(\rho). \tag{1.1}$$

Here, $\Phi$ is a 'fugacity' function, a continuum homogenization of the microscopic rate $g$ (cf. (2.4)). We note this equation also arises in the context of the zero-range system without disorder. (cf. Chapter 4 [17]).

However, at a point $x_j$ where $\beta_j = \alpha$, an atom evolves with a mass given by $\mathfrak{m}_j(t) = \big\{\lambda_j \Phi(\rho(t, x_j))\big\}^{1/\alpha}$, in terms of $\rho$. Such a statement is natural, given that under $\mu^N$, not far from $\mathscr{R}_{c_0}^N$, there are $O(N)$ particles at $k_{j,N}$. Differently, when $\beta_j > \alpha$, we will observe in the limit that there is an infinite mass at $x_j$. In diffusive scale, the boundary behavior at $k_{j,N}$ does not evolve much, leading to a macroscopic boundary condition $\rho(t, x_j) = c_0$ at $x_j$.

Taken together, the limit measure $\pi_t$ on $\mathbb{T}$ is in form

$$\pi_t = \rho(t, x)dx + \sum_{j:\beta_j = \alpha} \mathfrak{m}_j(t) \delta_{x_j}(dx)$$

for $t > 0$. At time $t = 0$, $\pi_t$ reduces to $\pi_0$ mentioned earlier. Moreover, the limit $\pi_t$ is characterized as the unique weak solution to a system $(\rho(t, x), \{\mathfrak{m}_j\})$; see Definition 4.4 and uniqueness Theorem 11.1. See Example 4.7, for a specific discussion when there is only one defect in the system. We mention in [16], under independent particles, that is when $g(n) \equiv n$ and $\alpha = 1$, the above hydrodynamic limit may be inferred when $\beta_j \equiv \alpha = 1$, with $\Phi(u) \equiv u$.

*Bounded rates $g$.* Our second result (Theorem 4.2) concerns the process with a bounded, rate function $g$, increasing say to level $g^* = 1$ in the limit as $k \uparrow \infty$. Slow sites $k$ that we consider will be those where the jump rate is $\frac{1}{\lambda N^{\beta_j}} g(n)$, when $\beta_j = 0$ and $\lambda > 1$, or $\beta_j < 0$, when there are $n$ particles at $k$. As before, with a finite number of slow sites $k_{j,N}$, there is a family of product invariant measures $\{\mathscr{R}_c^N\}$, but now with densities limited to $0 \leq \max_j \{\lambda_j N^{\beta_j} \vee 1\} \Phi(c) \leq g^*$. There is no non-trivial product invariant measure for $c$ above this level. In particular, to explain the focus on $\beta_j \leq 0$, if we would slow down with $\beta_j > 0$, there is no such non-trivial invariant measure, as the fugacity $\lambda_j N^{\beta_j} \Phi(c)$ will exceed $g_\infty$ for a finite $N$.

Phenomenologically, the behavior is different here than when $\lim_{n \uparrow \infty} g(n) = \infty$ in that the system and in particular the slow sites under $\mathscr{R}_c^N$ may have at most $O(N)$ particles on them. As before, between macroscopic defects $\{x_j \sim k_{j,N}/N\}$, the hydrodynamic limit satisfies (1.1). An atom of mass $\mathfrak{m}_j(t)$ may also form though at macroscopic site $x_j \sim k_{j,N}/N$ where $\beta_j = 0$ and $\lambda_j > 1$.

However, unlike before, this atomic mass may be evanescent: We show that, at such a defect $x_j$, the hydrodynamic density $\rho(t, x_j)$ satisfies a bound, $\lambda_j \Phi(\rho(t, x_j)) \leq g^*$. If the inequality is strict at time $t$, then necessarily $\mathfrak{m}_j(t) = 0$. Again, characterization of the limit $\pi_t = \rho(t,x)dx + \sum_j \mathfrak{m}_j(t)\delta_{x_j}(dx)$ is given through a weak formulation which is shown to have a unique solution; see Definition 4.6 and uniqueness Theorem 11.2.

A sufficient condition to not feel the defects $\{x_j\}$ in the limit would be to start the process from initial (local equilibrium) measures $\mu^N$ associated to $\pi_0(dx) = \rho_0(x)dx$, where $\|\rho_0\|_{L^\infty(\mathbb{T})} < \Phi^{-1}\big([\max_j\{\lambda_j\}]^{-1}\big)$, so that by say attractiveness $\rho(t,x)$ also satisfies this bound. On the other hand, a sufficient condition so that atom masses form is that the initial density $\rho_0(x) > \Phi^{-1}\big([\max_j\{\lambda_j\}]^{-1}\big)$ on $\mathbb{T}$: Indeed, if no atoms are formed, by the maximum principle, $\rho(t,x) \geq \Phi^{-1}\big([\max_j\{\lambda_j\}]^{-1}\big)$ on $\mathbb{T}$, a contradiction.

We mention, in the context of asymmetric zero-range processes in $d = 1$, a related but different phenomena is formulated in [18] (see also Section 3.3 in [4]). There, boundary behaviors near a slow site depend on the direction and an evolution of the mass of an atom there is also specified.

*Proof ideas for Theorem 4.1 and 4.2.* We now discuss ideas in the proofs. We use the general scheme of the 'entropy' method, discussed in [17], although there are several departures since the dynamics is not translation invariant. In particular, mixing in time estimates are used to make microscopic to macroscopic homogenizations in the 'bulk'. However, to capture the boundary effects and correspondences with macroscopic mass of atoms, we need to estimate the local behavior near defects, for which we develop a more refined argument.

Between macroscopic defects $x_j$ with $\beta_j > \alpha$, say with test function $G$ supported strictly away from these points, we take the stochastic differential

$$d\langle G, \pi_t^N\rangle = \frac{1}{N}\sum_{k\in\mathbb{T}_N} \Delta_N G\big(k/N\big) g_k(\eta_t(k))dt + dM^N(t).$$

Here, $g_k = \frac{1}{\lambda_j N^{-\beta_j}} g$ when $k = k_{j,N}$ and $g_k = g$ otherwise. Also, $M^N(t)$ is a martingale, which will vanish as $N \uparrow \infty$. We state a 'bulk' replacement in Lemma 8.4. The proof we give makes use of local replacements, which will be useful to deduce boundary behaviors at the defects.

We may replace the local function $g(\eta_t(k))$ at regular sites $k$ by $\Phi\big(\eta_t^{\theta N}\big)$ where $\eta^\ell$ is the $\ell$-window average $\frac{1}{2\ell+1}\sum_{|y-k|\leq \ell} \eta(y)$, by use of an entropy inequality and 'Rayleigh' estimate. The 'Rayleigh' estimate controls the difference between expectations of a local function under non-equilibrium and equilibrium measures in terms of a Dirichlet form and a term written in terms of the spectral gap of the process localized to an interval. No special bound is required–this gap needs only to be positive, or equivalently that the localized dynamics is ergodic. We remark, as a technical device, attractiveness is used to handle large densities, difficult to analyze otherwise, in the proofs of these local replacement, '1-block' Lemma 8.1 and '2-block' Lemma 8.2.

*Rates $g(n) \sim n^\alpha$.* In the setting where $g(n) \sim n^\alpha$ for $\alpha > 0$, at a defect $k_{j,N}$, when $\beta_j < \alpha$, under $\mu^N$, the number of particles $\eta_t(k_{j,N}) = O(N^{\beta/\alpha})$ is sublinear in $N$. Hence, with respect to the empirical measure, the term

$$N^{-1}\eta_t(k_{j,N}) = N^{-1}\eta_t(k_{j,N} \leq CN^{-1}\eta_t(k_{j,N}) = O(N^{-1+\beta/\alpha})$$

vanishes as $N \uparrow \infty$. In effect, we can ignore such defects in the continuum limit.

However, when $\beta_j \geq \alpha$, to deduce a non-trivial boundary condition, we consider the rate $g_{k_{j,N}}(\eta_t(k_{j,N})) = \frac{1}{\lambda_j N^{\beta_j}} g(\eta_t(k_{j,N})$ at the slow site $k_{j,N}$. To fix ideas, let us say $k_{j,N} = 0$, the origin. Under $\mu^N$, not far from the reversible $\mathscr{R}_{c_0}^N$ in terms of relative entropy, we show that this rate is close to the neighboring rate $g(\eta_t(1))$ say; see Lemma 9.1. By the local '1-block' estimate, we show that $g(\eta_t(1))$ is close to $\Phi(\eta_t^{\ell,+}(1))$, where $\eta_t^{\ell,+} = \frac{1}{\ell}\sum_{y=1}^{\ell} \eta_t(y)$ is the average in the $\ell$-block to the right; see Lemma 9.2. Now, by the local '2-block' bound, we have $\Phi(\eta_t^{\ell,+}(1))$ is close to the macroscopic quantity $\Phi((\eta_t^{\theta N,+}(1))$ for $\theta > 0$ small. Together, we conclude

$$\Phi(\eta_t^{\theta N,+}(1)) \sim g_0(\eta_t(0)), \tag{1.2}$$

a fundamental relation for us.

After establishment of tightness of the empirical measure trajectories, and absolute continuity estimates in Lemmas 6.1 and 7.1, we may consider a limit point $\pi$ on $\mathbb{T} \setminus \mathfrak{D}_s$ in the form $\pi = \rho(t,x)dx + \sum_{j:\beta_j=\alpha} \mathfrak{m}_j(t)$. From (1.2), one has on this limit point that

$$\Phi(\rho(t,0)) \sim g_0(\eta_t(0)).$$

When $\beta_j = \alpha$, we have

$$\Phi(\rho(t,0)) \sim g_0(\eta_t(0)) \sim \big(N^{-1}\eta_t(0)\big)^\alpha \sim \big(\mathfrak{m}_j(t)\big)^\alpha,$$

which is the boundary condition relating the atom strength to the local density in Theorem 4.1; see Lemma 7.4.

When $\beta_j > \alpha$, we may take the same passage to obtain

$$\Phi(\rho(t,0)) \sim g_0(\eta_t(0)) \sim \big(N^{-\beta_j/\alpha}\eta_t(0)\big)^\alpha.$$

But, in our time-scale, we show $N^{-\beta_j/\alpha}\eta_t(0)$ does not vary much from the time $t = 0$ quantity $N^{-\beta_j/\alpha}\eta_0(0)$. For instance, if $k_{j,N} = 0$ were the only defect in the system, by mass conservation,

$$|\eta_t(0) - \eta_0(0)| = |\sum_{k\neq 0} \eta_t(k) - \eta_0(k)|.$$

Starting under $\mu^N$, by use of the entropy inequality, $\sum_{k\neq 0}\eta_t(k) = O(N)$ for all $t \geq 0$. Hence,

$$N^{-\beta_j/\alpha}\big\{\eta_t(0) - \eta_0(0)\big\} = O(N^{\beta_j/\alpha - 1})$$

vanishes as $N \uparrow \infty$. Moreover, by a calculation with respect to the initial measure $\mu^N$, we may show that

$$\big(N^{\beta_j/\alpha}\eta_0(0)\big)^\alpha \sim \lambda_j \Phi(c_0).$$

As a consequence, we have

$$\Phi(\rho(t,0)) \sim \big(N^{\beta_j/\alpha}\eta_0(0)\big)^\alpha \sim \lambda_j\Phi(c_0),$$

yielding the boundary behavior in Theorem 4.1; see Lemma 7.2.

*Bounded rate $g$.* We now discuss the case of bounded rate function $g$. There are subtleties here with respect to boundary estimates, different than in the $n^\alpha$ rate setting, although the bulk hydrodynamic limit between defects $x_j \sim k_{j,N}/N$ follows the same procedure as above to derive the equation (1.1).

At a slow site $k_{j,N} = 0$, say at the origin, slowed down by $\lambda_j^{-1}$ (and $\beta_j = 0$), we deduce

$$\Phi(\rho(t,0)) \sim \frac{1}{\lambda_j} g(\eta_t(0)) \leq \frac{g^*}{\lambda_j}.$$

Hence, the density is restricted,

$$\rho(t,0) \leq \Phi^{-1}(\tfrac{g^*}{\lambda_j}).$$

When the restriction is strict, the intuition is that the particle mass is not high enough to impact much the variation of particle numbers at the slow site 0. However, when equality in the above relation holds, particles entering the slow site may stay there as the rate to depart is less than to enter. Hence, in this case, at atom at the slow site 0 may form to store excess mass while maintaining the boundary condition; see Lemma 7.5. We state in Theorem 4.2 that the hydrodynamic limit $\pi_t$ is in form $\pi_t = \rho(t,x)dx + \sum_{j:\lambda_j>1,\beta_j=0} \mathfrak{m}_j(t)$. Taken together, with the boundary prescriptions, we show $\pi_t$ solves uniquely the system in Definition 4.6; see also Theorem 11.2.

*On complements.* Although, we have considered system behaviors when starting from measures $\mu^N$ in a sense close to $\mathscr{R}^N_{c_0}$, one might ask about other initial conditions. For instance, we might start from measures where the density levels at 'super-slow' sites are not all $c_0$. Alternatively, the process could begin from initial states concentrated on $O(N)$ particles, as opposed to the super-linear in $N$ numbers in the $g(n) \sim n^\alpha$ setting. Different boundary structures, including reservoirs are also of interest. These and related concerns point toward natural complements to pursue in future work.

*Paper outline.* The plan of the paper is as follows: In Section 2 and 3, we introduce carefully the zero-range model with defects, their invariant measures, and the initial measures considered. In Section 4, we state results; see Section 4.7 for a discussion when there is only one defect in the system. In Section 5, we give the proof outline of the main results, Theorems 4.1 and 4.2, referring to estimates in the sequel. In Section 6, we discuss tightness of the empirical measures. In Section 7, we discuss properties of limit points, including importantly their boundary behaviors near macroscopic defects. In Section 8, we show local '1-block' and '2-block' replacements, and as a consequence 'bulk' replacement. In Section 9, we discuss replacements near the boundaries of defects needed to derive the macroscopic boundary conditions. In Sections 10 and 11, we derive energy estimates and prove uniqueness theorems for the weak formulations.

## 2. Model description

We will consider symmetric zero-range processes on the discrete torus $\mathbb{T}_N := \mathbb{Z}/(N\mathbb{Z}) = \{0,1,\dots,N-1\}$ with a finite number $n_0 \geq 0$ of defects located at $\mathfrak{D}_N \subset \mathbb{T}_N$. We will always assume that $N > n_0$ so that there is enough space in $\mathbb{T}_N$ for the defects.

More carefully, the structure of the defects is the following: Let $J$ be the index set $\{1,2,\dots,n_0\}$. For each $j \in J$, fix $(x_j,\beta_j,\lambda_j)$ such that $x_j \in \mathbb{T} := [0,1)$, $\beta_j \in \mathbb{R}$ and $\lambda_j \in (0,\infty)$. We will assume that all $x_j$'s are different, that is macroscopically separated. For each $j$, the point $x_j$ denotes the macroscopic location of a defect and $(\beta_j,\lambda_j)$ characterizes its strength. Let $\mathfrak{D} := \{x_j\}_{j\in J}$ be the set of all macroscopic defect locations. For each $j \in J$ and $N \in \mathbb{N}$, we now define $k_{j,N} = \lfloor x_j N \rfloor$ be the integer part of $x_j N$. Then, the set $\mathfrak{D}_N := \{k_{j,N}\}_{1\leq j\leq n_0}$. stands for the set of microscopic locations of the defects.

Let now $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. For each $N$, let $\Omega_N := \mathbb{N}_0^{\mathbb{T}_N}$ be the space of all particle configurations on $\mathbb{T}_N$. With respect to $\xi \in \Omega_N$, at a normal site $k \in \mathbb{T}_N \setminus \mathfrak{D}_N$, a particle jumps to neighboring sites $k \pm 1$ equally likely at rate $g(\xi(k))/\xi(k)$ where $\xi(k)$ is the number of particles at site $k$, and $g : \mathbb{N}_0 \to [0,\infty)$ is jump rate function. At a defect site $k = k_{j,N}$, the jump rate is $(\lambda_j N^{\beta_j})^{-1} g(\xi(k))/\xi(k)$. In particular, the site $k_{j,N}$ is a slow site if $\lambda_j N^{\beta_j} > 1$

and a fast site if $\lambda_j N^{\beta_j} < 1$. Let

$$g_{k,N}(\cdot) = \begin{cases} g(\cdot) & k \in \mathbb{T}_N \setminus \mathfrak{D}_N, \\ \dfrac{g(\cdot)}{\lambda_j N^{\beta_j}} & k = k_{j,N},\ j \in J. \end{cases}$$

This zero-range process is a Markov process $\xi_t$ with the generator

$$L_N f(\xi) = \sum_{k \in \mathbb{T}_N} \Big\{ g_{k,N}(\xi(k))\big(f(\xi^{k,k+1}) - f(\xi)\big) + g_{k,N}(\xi(k))\big(f(\xi^{k,k-1}) - f(\xi)\big)\Big\} \tag{2.1}$$

where

$$\xi^{x,y}(k) = \begin{cases} \xi(x) - 1 & k = x, \\ \xi(y) + 1 & k = y, \\ \xi(k) & k \neq x, y. \end{cases} \tag{2.2}$$

So that the process is irreducible, we will assume that the jump rate function $g$ is such that $g(n) = 0$ exactly when $n = 0$. We will also assume the following condition.

**Condition 2.1.** *The jump rate function $g(\cdot)$ satisfies*

(1) *Lipschitz: there exists $g^* > 0$ such that $|g(n+1) - g(n)| \leq g^*$ for all $n \in \mathbb{N}_0$.*
(2) *Power Interaction: there exists $\alpha \in [0,1]$ such that $g(n) \sim n^\alpha$, that is*
$$\lim_{n\to\infty} \frac{g(n)}{n^\alpha} = 1.$$
(3) *Monotonicity: $g(n) \leq g(n+1)$ for all $n \in \mathbb{N}_0$.*

When the constant $\alpha = 0$, in the power interaction condition, we note the function $g(n)$ increases to $g_\infty := 1$ as $n \to \infty$. We will refer to this case as 'g is of bounded type' or simply, $g$ is *bounded*. On the other hand, when $\alpha \in (0,1]$, we will say 'g is of $n^\alpha$ type'.

We remark that, when $g(n) \equiv n$, the dynamics in non-interactive, a superposition of independent random walks. Moreover, the assumption that $g$ is increasing, an 'attractive' dynamics assumption, allows use of the 'basic coupling' in our proofs. For more discussion on this point, see Section 3.2.

Our goal in this work is to study hydrodynamic limits of the dynamics generated by $L_N$. In particular, in the limit macroscopic flow, there will be different behaviors at the defect locations depending on the associated strength parameters To prepare for these statements, it will be helpful to introduce a partition on the index set $J = J_s \cup J_c \cup J_b$. Here, the subscripts $s$, $c$, and $b$ stand for super-critical or 'super-slow', critical, and sub-critical respectively.

There will be different partitions of $J$ depending on whether $g$ is of $n^\alpha$ type or bounded. When $g$ is of $n^\alpha$ type, we let

- $J_s := \{j \in J : \beta_j > \alpha\}$,
- $J_c := \{j \in J : \beta_j = \alpha\}$, and
- $J_b := \{j \in J : \beta_j < \alpha\}$.

When $g$ is bounded, we take

- $J_s := \{j \in J : \beta_j > 0\}$,
- $J_c := \{j \in J : \beta_j = 0,\ \lambda_j > 1\}$, and
- $J_b := \{j \in J : \beta_j < 0 \text{ or } \beta = 0 \text{ with } \lambda_j < 1\}$.

In terms of the partition of $J$, the sets $\mathfrak{D}$ and $\mathfrak{D}_N$, the macroscopic and microscopic locations of the defects, are then divided into corresponding subsets. For example, we have $\mathfrak{D}_s := \{x_j \in \mathfrak{D} : j \in J_s\}$ and $\mathfrak{D}_{s,N} := \{k_{j,N} \in \mathfrak{D}_N : j \in J_s\}$. The sets $\mathfrak{D}_c$, $\mathfrak{D}_b$, $\mathfrak{D}_{c,N}$, and $\mathfrak{D}_{b,N}$ are defined in the same fashion.

2.1. **Invariant measures.** The construction of invariant measures under $L_N$ is based on $\{\mathcal{P}_\phi\}$, a family of Poisson-like distributions indexed by 'fugacities' $\phi$. In order to define $\mathcal{P}_\phi$, we first introduce the partition function:

$$Z(\phi) := \sum_{n=0}^{\infty} \frac{\phi^n}{g(n)!}.$$

Let $r_g = \lim_{n\to\infty} g(n)$ be the convergence radius of $Z(\cdot)$. In particular, when $g$ is of $n^\alpha$ type, we have $r_g = \infty$, namely, the "FEM" condition (cf. p. 69, [17]) is satisfied. When $g$ is bounded, $r_g = g_\infty = 1$. In either case, it holds that $\lim_{\phi\to r_g} Z(\phi) = \infty$.

For each $\phi \in [0, r_g)$, define $\mathcal{P}_\phi$ by

$$\mathcal{P}_\phi(n) = \frac{1}{Z(\phi)} \frac{\phi^n}{g(n)!}, \quad \text{for } n \geq 0. \tag{2.3}$$

Here, $g(0)! := 1$ and $g(n)! := \prod_{k=1}^n g(k)$ for $n \geq 1$. Let $R(\phi) = E_{\mathcal{P}_\phi}[X]$, where $X(n) = n$, be the mean of the distribution $\mathcal{P}_\phi$. A direct computation yields that $R'(\phi) > 0$, $R(0) = 0$, and $\lim_{\phi\to r_g} R(\phi) = \infty$. Since $R$ is strictly increasing, it has an inverse, denoted by $\Phi : [0,\infty) \mapsto [0, r_g)$. We may parametrize the family of distributions $\mathcal{P}_\phi$ by their means: For $\rho \geq 0$, let $\mathcal{Q}_\rho = \mathcal{P}_{\Phi(\rho)}$, so that $E_{\mathcal{Q}_\rho}[X] = E_{\mathcal{P}_{\Phi(\rho)}}[X] = R(\Phi(\rho)) = \rho$. Here and in the following, with respect to a given probability measure $\mu$, we denote by $E_\mu$ and $\mathrm{Var}_\mu$ its expectation and variance.

A straightforward computation yields that $E_{\mathcal{P}_\phi}[g(X)] = \phi$ for $\phi \geq 0$. Thus,

$$\Phi(\rho) = E_{\mathcal{P}_{\Phi(\rho)}}[g(X)] = E_{\mathcal{Q}_\rho}[g(X)] \ \ , \quad \rho \ \geq \ 0 \ . \tag{2.4}$$

As $g(n) \leq g^* n$, we have that $\Phi(\rho) \leq g^*\rho$. A simple computation yields that $\Phi'(\rho) = \Phi(\rho)/\sigma^2(\rho)$ where $\sigma^2(\rho)$ is the variance of $X$ under $\mathcal{Q}_\rho$. Under our assumptions on $g$, in fact, it holds that $0 \leq \Phi'(\rho) \leq g^*$ for all $\rho \geq 0$ (cf. p. 33, [17]). In particular, $\Phi \in C^1[0,\infty)$ is an increasing function with a uniformly bounded derivative. We note, in the case $g(n) \equiv n$, that $\Phi(\rho) \equiv \rho$ and $\mathcal{P}_\phi$ is a Poisson measure with mean $\phi$.

We now introduce the invariant measures. For each $N$, let

$$q_N = \max\{1, \lambda_j N^{\beta_j} | j \in J\}.$$

For $c$ so that $\Phi(c) \in [0, r_g/q_N)$, denote by $\mathscr{R}_c^N$ the product measure on $\Omega_N$ whose marginals are given by

$$\mathscr{R}_c^N(\xi(k) = n) = \begin{cases} \mathcal{P}_{\Phi(c)}(n) & \text{for } k \in \mathbb{T}_N \setminus \mathfrak{D}_N \text{ and } n \geq 0, \\ \mathcal{P}_{\lambda_j N^{\beta_j}\Phi(c)}(n) & \text{for } k = k_{j,N},\ j \in J, \text{ and } n \geq 0. \end{cases} \tag{2.5}$$

Notice that the condition $\Phi(c) \in [0, r_g/q_N)$ is needed since the distributions $\{\mathcal{P}_\phi\}$ are defined for $\phi \in [0, r_g)$. When $g$ is of $n^\alpha$ type, as $r_g = \infty$, we have that $\{\mathscr{R}_c^N\}$ are defined for all $c \in [0,\infty)$. However, when $g$ is bounde, we have $r_g = g_\infty = 1$, and thus the measures $\{\mathscr{R}_c^N\}$ are defined only for $c \in [0, R(1/q_N))$ in this case.

With $\mathscr{R}_c^N$ defined, it is straightforward (cf. [1], [10]) to check the following lemma.

**Lemma 2.2.** *For $c$ so that $\Phi(c) \in [0, r_g/q_N)$, $\mathscr{R}_c^N$ is invariant and reversible with respect to the generator $L_N$ in* (2.1).

2.2. **Static limit.** Before studying the hydrodynamic limits, it will be useful to understand behavior under an invariant measure $\mathscr{R}_c^N$. For a configuration $\xi \in \Omega_N$, define the associated scaled mass empirical measure:

$$\widehat{\pi}^N(dx) := \frac{1}{N} \sum_{k \in \mathbb{T}_N} \xi(k)\delta_{k/N}(dx). \tag{2.6}$$

In this formulation, each particle has mass $N^{-1}$. Here and in the sequel, $\delta_z$ refers to a delta point mass at $z$.

For a test function $G \in C(\mathbb{T})$, let

$$\langle G, \widehat{\pi}^N \rangle := N^{-1} \sum_{k \in \mathbb{T}_N} \xi(k) G(k/N).$$

More generally, the notation $\langle G, \mu \rangle := \int G d\mu$. We now compute the limit of $\langle G, \widehat{\pi}^N \rangle$ with respect to a sequence of invariant measures $\mathscr{R}_c^N$ with $c$ fixed as $N \to \infty$.

We assume first $g$ is of $n^\alpha$ type and $c > 0$. Because of the product structure of $\mathscr{R}_c^N$, $\{\xi(k)\}_{k \in \mathbb{T}_N}$ are independent and have a commmon marginal $\mathcal{P}_{\Phi(c)}$ for all $k \neq \mathfrak{D}_N$. As $\mathcal{P}_{\Phi(c)}$ has expectation $c$ and finite variance, we have $N^{-1} \sum_{k \notin \mathfrak{D}_N} \xi(k) G(k/N)$ converges in probability to $\int_{\mathbb{T}} G(x) c\, dx$ as $N \to \infty$.

It remains to investigate the behavior of $N^{-1}\xi(k)$ for $k = k_{j,N} \in \mathfrak{D}_N$. As $\xi(k_{j,N})$ has distribution $\mathcal{P}_{\lambda_j N^{\beta_j} \Phi(c)}$, by the later Lemma 2.5, for all $\beta_j > 0$

$$E_{\mathscr{R}_c^N}[\xi(k_{j,N})] \sim (\lambda_j \Phi(c))^{1/\alpha} N^{\beta_j/\alpha}, \quad \text{and} \quad \mathrm{Var}_{\mathscr{R}_c^N}[\xi(k_{j,N})] = o(N^{2\beta_j/\alpha}).$$

Then, according to the value of $\beta_j$, there are three different types of behaviors at $k \in \mathfrak{D}_N$:

(1) if $\beta_j < \alpha$, $N^{-1}\xi(k_{j,N}) \to 0$ in probability;
(2) if $\beta_j = \alpha$, $N^{-1}\xi(k_{j,N}) \to (\lambda_j \Phi(c))^{1/\alpha}$ in probability;
(3) if $\beta_j > \alpha$, $N^{-\beta_j/\alpha}\xi(k_{j,N}) \to (\lambda_j \Phi(c))^{1/\alpha}$ in probability.

In other words, the defect site $k_{j,N}$ becomes macroscopically invisible when $\beta_j < \alpha$ as typically it contains $o(N)$ number of partilces and each particle has mass $N^{-1}$. In the case $\beta_j = \alpha$, typically number of particles at $k_{j,N}$ is of order $N$ and a delta mass of magnitude $(\lambda_j \Phi(c))^{1/\alpha}$ emerges. When the site is super-slow, that is $\beta_j > \alpha$, the particle number at $k_{j,N}$ is of order $N^{\beta_j/\alpha}$ which corresponding to an infinite macroscopic mass. Recall, the partition $J = J_b \cup J_c \cup J_s$ in Section 2 matches with this classification of the $\beta_j$'s.

As the macroscopical mass at $x_j \in \mathfrak{D}_s$ explodes, to consider the remaining mass, we define microscopic empirical measures which exclude the super-critical defect set $\mathfrak{D}_{s,N}$:

$$\pi^N(dx) := \frac{1}{N} \sum_{k \in \mathbb{T}_N \setminus \mathfrak{D}_{s,N}} \xi(k)\delta_{k/N}(dx). \tag{2.7}$$

Then, we may summarize the above discussion as follows.

**Proposition 2.3.** *Assume $g$ is of $n^\alpha$ type. Then, for any $G \in C(\mathbb{T})$, $c \geq 0$, and $\delta > 0$:*

$$\lim_{N \to \infty} \mathscr{R}_c^N \left[ \left| \langle G, \pi^N \rangle - \langle G, \pi \rangle \right| > \delta \right] = 0$$

*where $\pi(dx) = c\, dx + \sum_{j \in J_c} (\lambda_j \Phi(c))^{1/\alpha} \delta_{x_j}(dx)$. Moreover, for all $j \in \mathfrak{D}_s$, we have*

$$\lim_{N \to \infty} \mathscr{R}_c^N \left[ \left| N^{-\beta_j/\alpha}\xi(k_{j,N}) - (\lambda_j \Phi(c))^{1/\alpha} \right| > \delta \right] = 0.$$

Now we turn to the case when $g$ is bounded. Recall, in this case, $\mathscr{R}_c^N$ is defined for $c$ such that $\Phi(c) \in [0, 1/q_N)$ where $q_N = \max\{1, \lambda_j N^{\beta_j} | j \in J\}$. Therefore, when the set $J_s = \{j \in J : \beta_j > 0\}$ is not empty, the only possible constant $c$ allowed is $c = 0$. Hence, we will assume $J_s = \emptyset$. When $J_c \neq \emptyset$, we will take $q_N = \lambda_{\max} := \max_{j \in J_c}\{\lambda_j\}$; otherwise, when also $J_c = \emptyset$, we take $q_N = 1$.

For $c \in [0, R(1/\lambda_{\max}))$, we also have $N^{-1}\sum_{k \notin \mathfrak{D}_N} \xi(k) G(k/N)$ converges in probability to $\int_{\mathbb{T}} G(x) c\, dx$ as $N \to \infty$. Given $\xi(k)$ has finite expectation and variance for all $k \in \mathfrak{D}_N$, the following is easily obtained.

**Proposition 2.4.** *Assume $g$ is bounded and $J_s = \emptyset$. Then, the domain for the parameter $c$ in $\mathscr{R}_c^N$ is $c \in [0, R(1/q_N))$. Also, for any $G \in C(\mathbb{T})$ and $\delta > 0$, we have*

$$\lim_{N\to\infty} \mathscr{R}_c^N \Big[ \Big| \langle G, \pi^N \rangle - \int_{\mathbb{T}} G(x) c\, dx \Big| > \delta \Big] = 0.$$

We finish this section with a technical lemma used in Proposition 2.3.

**Lemma 2.5.** *Assume $g(j) \sim j^\alpha$ for some $\alpha \in (0, 1]$. For each $\varphi > 0$, let $X$ be a random variable with distribution $\mathcal{P}_\varphi$ (cf. (2.3)). Then, for each $n \in \mathbb{N}$, $E[X^n] \sim \varphi^{n/\alpha}$ as $\varphi \to \infty$. As a result, $\ln Z(\varphi) \sim \alpha \varphi^{1/\alpha}$ and $\mathrm{Var}[X] = o\big(\varphi^{2/\alpha}\big)$ as $\varphi \to \infty$.*

*Proof.* We first show $E[X^n] \sim \varphi^{n/\alpha}$ for all $n \in \mathbb{N}$. To this end, let us assume for now

$$\sum_{k=1}^{\infty} \frac{k^{n\alpha} \varphi^k}{g(k)!} \sim \sum_{j=n}^{\infty} \frac{\varphi^k}{g(j-n)!}. \tag{2.8}$$

Let $Y = X^\alpha$. Then,

$$E[Y^n] = \frac{1}{Z(\varphi)} \sum_{k=1}^{\infty} \frac{k^{n\alpha}\varphi^k}{g(k)!} \sim \frac{1}{Z(\varphi)} \sum_{k=n}^{\infty} \frac{\varphi^k}{g(k-n)!} = \varphi^n.$$

As $\alpha \in (0, 1]$, we may find $p \in [0, 1)$ and $l \in \mathbb{N}$ such that $\alpha^{-1} = p + (1-p)l$. Also, since $E[X^n] = E[Y^{n/\alpha}]$, by Jensen's and Hölder's inequalities,

$$E[Y^n]^{1/\alpha} \le E[X^n] \le E[Y^n]^p E[Y^{nl}]^{1-p}.$$

Since $E[Y^n] \sim \varphi^n$ and $E[Y^{nl}] \sim \varphi^{nl}$, we obtain $E[X^n] \sim \varphi^{n/\alpha}$.

For the limit behavior of $E[X^n]$, it remains to show the claim (2.8). As $g(k) \sim k^\alpha$, for any $A > 0$, we may find $\lambda_1 = \lambda_1(A)$ and $\lambda_2 = \lambda_2(A)$, such that $\lambda_1 k^\alpha \le g(k) \le \lambda_2 k^\alpha$ for all $k \ge A$ and $\lim_{A\to\infty} \lambda_1 = \lim_{A\to\infty} \lambda_2 = 1$. Then, for all $k \ge A + n$,

$$\lambda_2^{-n} \le \frac{k^{n\alpha}}{\prod_{k-n<j\le k} g(j)} \le \Big(\frac{A+n}{A}\Big)^n \lambda_1^{-n}.$$

Therefore,

$$\lambda_2^{-n} \sum_{k=A+n}^{\infty} \frac{\varphi^k}{g(k-n)!} \le \sum_{k=A+n}^{\infty} \frac{k^{n\alpha}\varphi^k}{g(k)!} \le \Big(\frac{A+n}{A}\Big)^n \lambda_1^{-n} \sum_{k=A+n}^{\infty} \frac{\varphi^k}{g(k-n)!}$$

Notice that, if $\varphi$ is sent to infinity in the above display, we may replace $\sum_{k\ge A+n}$ by either $\sum_{k\ge 0}$ or $\sum_{k\ge n}$. Then the claim (2.8) follows from taking $\varphi \to \infty$ and then $A \to \infty$.

We have shown $E[X^n] \sim \varphi^{n/\alpha}$ for all $n \in \mathbb{N}$. Then, it follows that $\mathrm{Var}[X] = o\big(\varphi^{2/\alpha}\big)$ as $E[X^2] \sim E[X]^2 \sim \varphi^{2/\alpha}$. To prove $\ln Z(\varphi) \sim \alpha\varphi^{1/\alpha}$, notice $\dfrac{d}{d\varphi} \ln Z(\varphi) = \varphi^{-1} E[X] \sim \varphi^{1/\alpha - 1}$ and then apply L'Hospital's rule, to finish the argument. □

## 3. Initial measures

In this section, we specify the assumptions on the initial measures $\{\mu^N\}$ we use to start our dynamics. Roughly speaking, $\{\mu^N\}$ should be associated with a macroscopic profile which gives the initial condition for the hydrodynamic limit. We will also require $\mu^N$ to possess certain relative entropy estimates and to be stochastically bounded with respect to invariant measures.

To specify these conditions, recall, for $\mu$, $\nu$, two probability measures on $\Omega_N$, we say that $\mu \leq \nu$, that is $\mu$ is stochastically bounded by $\nu$, if for all $f : \Omega_N \mapsto \mathbb{R}$ coordinately increasing, we have $E_\mu(f) \leq E_\nu(f)$. Fix also $\pi$, a nonnegative measure on $\mathbb{T}$, such that

$$\pi(dx) = \rho_0(x)dx + \sum_{j\in J_c} \mathfrak{m}_{0,j}\, \delta_{x_j}(dx) \tag{3.1}$$

where $\rho_0(x) \in L^1(\mathbb{T})$ and $\mathfrak{m}_{j,0} \geq 0$ for $j \in J_c$. Recall also, the empirical measure defined in (2.7).

Throughout this work, we will assume the following on the sequence of initial measures $\{\mu^N\}_{N\in\mathbb{N}}$ on $\Omega_N$.

**Condition 3.1.** *The following hold:*

(1) $\{\mu^N\}_{N\in\mathbb{N}}$ *has macroscopic profile* $\pi$ *on* $\mathbb{T}\setminus\mathfrak{D}_s$*, i.e. for all* $G(x)\in C(\mathbb{T})$ *and* $\delta>0$*,*

$$\lim_{N\to\infty} \mu^N\Big[\, \Big|\langle G, \pi^N\rangle - \langle G,\pi\rangle\Big| > \delta\,\Big] = 0.$$

(2) *There exists* $c_0>0$ *such that the relative entropy of* $\mu^N$ *with respect to* $\mathscr{R}^N_{c_0}$ *is of order* $N$*: Let* $f_N := d\mu^N/d\mathscr{R}^N_{c_0}$*. Then,* $H(\mu^N|\mathscr{R}^N_{c_0}) := \int f_N \ln f_N d\mathscr{R}^N_{c_0} = O(N)$*.*

(3) *If* $g$ *is of* $n^\alpha$ *type, then there exists* $c'$ *such that* $\mu^N$ *is stochastically bounded by* $\mathscr{R}^N_{c'}$ *for all* $N$*.*

(3′) *If* $g$ *is bounded, then there exists* $c'$ *such that, when restricted on* $\mathbb{T}_N\setminus\mathfrak{D}_{c,N}$*,* $\mu^N$ *is stochastically bounded by* $\kappa^N_{c'}$*, the product measure on* $\Omega_N$ *with marginal* $\mathcal{P}_{\Phi(c')}$ *for* $k\in\mathbb{T}_N\setminus\mathfrak{D}_{b,N}$ *and* $\mathcal{P}_{\lambda_j N^{\beta_j}\Phi(c')}$ *for* $k=k_{j,N}\in\mathfrak{D}_{b,N}$*.*

We comment that item (3) is sufficiently general in the $n^\alpha$ setting to allow $\{\mu^N\}$ to be associated with profiles of the form $\pi_0(dx) = \rho_0(x)dx + \sum_{j\in J_c}\mathfrak{m}_{0,j}\delta_{x_j}(dx)$, for densities $\rho_0$ and masses $\{\mathfrak{m}_{0,j}\}$, as demonstrated in Section 3.3. In the bounded $g$ case, however, if $\{\mu^N\}$ satisfied item (3), then $O(N)$ accumulations at points $\mathfrak{D}_{c,N}$ would not be allowed. In this case, the only profiles allowed would be of form $\pi_0(dx) = \rho_0(x)dx$, where $\|\rho_0\|_{L^\infty} \leq \min_{j\in J_c}\frac{1}{\lambda_j}$ (cf. Section 3.2). As mentioned in the introduction, hydrodynamic evolution from such profiles would not see any defects, by the maximum principle.

In this context, item (3′) is formulated on $\{\mu^N\}$ so that $O(N)$ accumulations are possible on $\mathfrak{D}_{c,N}$ as well as later point masses on $\mathfrak{D}_c$ in the hydrodynamic evolution.

In fact, conditions (3) and (3′) can be made to accommodate a larger class of initial measures $\{\mu^N\}$. For example, one may remove stochastic bounded assumptions on coordinates in $\mathfrak{D}_{b,N}$. We have however choosen to state the (3) and (3′) in the forms given to streamline arguments and avoid more piecemeal calculations.

In the rest of this section, we make remarks on the relative entropy bound, discuss the use of attractiveness, and also provides a large class of examples of $\mu^N$ which satisfy Condition 3.1.

### 3.1. **Relative entropy.**

We first comment on the $\mu^N$-particle numbers in the system. Denote $H_N := H(\mu^N|\mathscr{R}^N_c) = O(N)$.

Assume that $g$ is of $n^\alpha$ type. By the entropy inequality $E_\mu[f] \le H(\mu|\nu) + \ln E_\nu[e^f]$ (cf. p.338 [17]), $H_N = O(N)$ implies that

$$E_{\mu^N}\Big[\sum_{k\notin\mathfrak{D}_{s,N}} \xi(k)\Big] \le H(\mu^N|\mathscr{R}^N_{c_0}) + \ln E_{\mathscr{R}^N_{c_0}}\big[e^{\sum_{k\notin\mathfrak{D}_{s,N}}\xi(k)}\big].$$

As $\mathscr{R}^N_{c_0}$ is product measure, the ln term is written as $\sum_{k\notin\mathfrak{D}_{s,N}} \ln E_{\mathscr{R}^N_{c_0}}[e^{\xi(k)}]$ which is $O(N)$ by Lemma 2.5. Thus, the condition $H_N = O(N)$ allows initially $O(N)$ particles on the sites $\mathbb{T}_N \setminus \mathfrak{D}_{s,N}$. On the other hand, at a super-slow site $k_{j,N} \in \mathfrak{D}_{s,N}$, the particle number is typically of order $O(N^{\beta_j/\alpha})$. Then, because $H_N = O(N)$, with respect to $\mu^N$, $N^{-\beta_j/\alpha}\xi(k_{j,N})$ converges in probability to $(\lambda_j\Phi(c_0))^{1/\alpha}$(cf. (7.4)).

Let $\{\xi_t\}_{t\ge 0}$ be the random process generated by $L_N$ with initial measure $\mu^N$ and let $\mu^N_t$ be the distribution of $\xi_t$. Notice that the entropy does not increase in $t$, that is $H(\mu^N_t|\mathscr{R}^N_{c_0}) \le H_N$ (cf. pp 340, [17]). Therefore, the net exchange of particle numbers between super-slow sites $\mathfrak{D}_{s,N}$ and the rest of the system is on order $N$ and the total particle number on $\mathbb{T}_N \setminus \mathfrak{D}_{s,N}$ remains $O(N)$ for all time. We will describe the macroscopic evolution of the particles on $\mathbb{T}_N \setminus \mathfrak{D}_{s,N}$ by our later hydrodynamic limit result.

We now turn to case when $g$ is bounded. As remarked in Section 2.2, the existence of $\{\mathscr{R}^N_c\}_{N\in\mathbb{N}}$ for $c > 0$ requires that $J_s = \emptyset$, that is $\mathfrak{D}_s = \emptyset$. Therefore, given the assumption $H_N = O(N)$, the following condition makes sense.

**Condition 3.2.** *When $g$ is bounded, we assume $J_s = \emptyset$.*

Hence, by the previous entropy inequality discussion, it follows that the total number of particles in $\mathbb{T}_N$ is of order $N$ for all time.

3.2. **Basic coupling.** We now discuss the use of the stochastic boundedness assumption (3) and (3′) in Condition 3.1. Since $g(j)$ is an increasing function in $j$, the dynamics generated by $L_N$ is 'attractive': if initially $\xi_0$ is distributed according to measures $\mu \le \nu$, then we have $\mu_t \le \nu_t$ for all $t \ge 0$ where $\mu_t$ and $\nu_t$ are distributions of $\xi_t$ (cf. [1], Chapter II in [21]).

In the case when $g(j)$ of $j^\alpha$ type, as $\mathscr{R}^N_{c'}$ is invariant, the assumption $\mu^N \le \mathscr{R}^N_{c'}$ implies that $\mu^N_t \le \mathscr{R}^N_{c'}$ for all $t \ge 0$. However, when $g$ is bounded, the domain for $c$ in $\mathscr{R}^N_c$ is $c < R(1/\lambda_{\max})$ (cf. Proposition 2.4). Then, an assumption $\mu^N \le \mathscr{R}^N_{c'}$ would imply that $\pi$, the macroscopic profile associated to $\mu^N$, is $\pi(dx) = \rho_0(x)dx$ with $\|\rho_0\|_\infty < R(1/\lambda_{\max}) < \infty$. To accommodate more initial profiles $\pi$ (and observe more involved limit evolutions), we have assumed (3′) in Condition 3.1 instead, that is, for all coordinately increasing $f : \Omega_N \mapsto \mathbb{R}$ depending only on $\{\xi(k)\}_{k\notin\mathfrak{D}_N}$, it holds $E_{\mu^N}[f(\xi)] \le E_{\kappa^N_{c'}}[f(\xi)]$.

We now illustrate how we will use attractiveness under assumption (3′). Instead of an evolution with respect to particle numbers in $\mathbb{N}_0$ and configurations in $\Omega_N^{\mathbb{T}_N}$, we may consider a dynamics corresponding to $\overline{\mathbb{N}}_0 := \mathbb{N}_0 \cup \{\infty\}$ and $\overline{\Omega}_N := \overline{\mathbb{N}}_0^{\mathbb{T}_N}$.

Recall the constant $c'$ in (3′). Define $\overline{g}_{k,N}(\cdot)$ by $\overline{g}_{k,N}(n) = g_{k,N}(n)$ for $n \in \mathbb{N}_0$ and $\overline{g}_{k,N}(\infty) = \Phi(c')$. Consider now the following generator on $\overline{\Omega}_N$

$$\overline{L}_N f(\xi) = \sum_{k\in\mathbb{T}_N} \Big\{\overline{g}_{k,N}(\xi(k))\big(f(\xi^{k,k+1}) - f(\xi)\big) + \overline{g}_{k,N}(\xi(k))\big(f(\xi^{k,k-1}) - f(\xi)\big)\Big\}.$$

Here, $\xi^{x,y}$ is defined as in (2.2) by using the convention $\infty \pm 1 = \infty$. When starting with configurations in $\Omega_N$, $\overline{L}_N$ coincides with $L_N$. Once a site starts with $\infty$ particles, it will serve as a 'reservoir' which pumps particles into its neighbors at the rate of $\Phi(c')$. Let $\delta_\infty$ be the Dirac measure on the extended number $\infty$. Define $\overline{\kappa}^N_{c'}$ as the product measure on $\overline{\Omega}_N$

that coincides with $\kappa_{c'}^N$ on $k \notin \mathfrak{D}_{c,N}$ and has marginal $\delta_\infty$ for $k \in \mathfrak{D}_{c,N}$. It is straightforward to check that $\kappa_{c'}^N$ is invariant under $\overline{L}_N$. Also, we have that $\mu^N \leq \overline{\kappa}_{c'}^N$ by (3′) in Condition 3.1. As attractiveness and the basic coupling is still in effect for $\overline{L}_N$, we obtain the later time $t$ distribution $\mu_t^N \leq \overline{\kappa}_{c'}^N$ for any $t \geq 0$.

Finally, as a general remark, we make technical use of 'attractiveness' most essentially in the cutoff of large densities in the local replacements (Lemma 8.1 and Lemma 8.2), and in proving the limit mass profile has $L^2$ absolutely continuous part (Lemma 7.1). The latter property is needed in the proof of uniqueness of weak solutions (Theorem 11.1 and 11.2).

3.3. **Local equilibria.** We now give explicit examples of initial measures that satisfy the Condition 3.1. These examples will be denoted by $\mu_{\text{le}}^N$ as they are related with the usual 'local equilibrium' measures in setting without defects (cf. [17]).

Let $\pi$ be as in (3.1) with $\rho_0(x) \in L^\infty(\mathbb{T})$ and $\mathfrak{m}_{0,j} \geq 0$ for each $j \in J_c$. For each $k \in \mathbb{T}_N$, define

$$\rho_{k,N} = N \int_{(k-1)/N}^{k/N} \rho_0(x)dx.$$

We will construct $\{\mu^N\}$ separately for $g$ of $n^\alpha$ and bounded types. Consider first $g$ of $n^\alpha$ type. Fix $c_0 > 0$ and define

$$\varphi_{k,N} = \begin{cases} \Phi(\rho_{k,N}) & k \in \mathbb{T}_N \setminus \mathfrak{D}_N \\ 0 & k = k_{j,N} \in \mathfrak{D}_{b,N} \\ (N\mathfrak{m}_{0,j})^\alpha & k = k_{j,N} \in \mathfrak{D}_{c,N} \\ \lambda_j N^{\beta_j} \Phi(c_0) & k = k_{j,N} \in \mathfrak{D}_{s,N} \end{cases}$$

For each $N \in \mathbb{N}$, let $\mu_{\text{le}}^N$ be the product measure on $\Omega_N$ with marginals given by

$$\mu_{\text{le}}^N(\xi(k) = n) \;=\; \mathcal{P}_{\varphi_{k,N}}(n), \quad \text{for } k \in \mathbb{T}_N\,,\, n \geq 0.$$

**Lemma 3.3.** *Suppose $g$ is of $n^\alpha$ type. Then $\mu^N = \mu_{le}^N$ satisfies Condition 3.1*

*Proof.* Let $c'$ be such that $\Phi(c') = \max\{\Phi\left(\|\rho_0\|_\infty\right), (\mathfrak{m}_{0,j})^\alpha, \Phi(c_0)\}_{j \in J_c}$. As $\mathcal{P}_{\phi_1} \leq \mathcal{P}_{\phi_2}$ if $\phi_1 \leq \phi_2$ (cf. [21], pp. 32), we have that the product measure $\mu_{\text{le}}^N$ is stochastically bounded by $\mathscr{R}_{c'}^N$ for all $N$. As $G$ is uniformly continuous, that $\mu_{\text{le}}^N$ is associated with the given macroscopic profile $\pi$ holds straightforwardly from Chebychev inequality and Lemma 2.5.

It remains to check the desired entropy bound $H(\mu_{\text{lc}}^N|\mathscr{R}_{c_0}^N) = O(N)$. Let $\varphi = \Phi(c_0)$. We compute

$$H(\mu_{\text{le}}^N|\mathscr{R}_{c_0}^N) = \sum_{j\in J} H(\mathcal{P}_{\varphi_{k_{j,N},N}}|\mathcal{P}_{\lambda_j N^{\beta_j}\varphi}) + \sum_{k\in\mathbb{T}_N\setminus\mathfrak{D}_N} H(\mathcal{P}_{\varphi_{k,N}}|\mathcal{P}_\varphi).$$

Note that, for any $\phi_1 \geq 0$ and $\phi_2 > 0$,

$$H(\mathcal{P}_{\phi_1}|\mathcal{P}_{\phi_2}) = \ln\frac{\phi_1}{\phi_2} E_{\mathcal{P}_{\phi_1}}[X] + \ln\frac{Z(\phi_2)}{Z(\phi_1)}$$

where $X$ is defined as $X(n) = n$. We also adopt $0 \ln 0 = 0$ in the case $\phi_1 = 0$. By Lemma 2.5, we conclude the desired relative entropy bound $H(\mu_{\text{le}}^N|\mathscr{R}_{c_0}^N) = O(N)$. □

We now consider the $g$ bounded case. Note that we have $J_s = \emptyset$; see Condition 3.2. Let $\tilde{\varphi}_{k,N} = \Phi(\rho_{k,N})$ for $k \in \mathbb{T}_N \setminus \mathfrak{D}_N$ and $\tilde{\varphi}_{k,N} = 0$ for $k \in \mathfrak{D}_{b,N}$. Also, let $\mathcal{P}'_\lambda$ denote the Poisson distribution with mean $\lambda$.

We define $\tilde{\mu}^N_{\text{le}}$ as the product measure with marginals $\mathcal{P}_{\tilde{\varphi}_{k,N}}$ on sites $k \notin \mathfrak{D}_c$ and $\mathcal{P}'_{\mathfrak{m}_{0,j}N}$ for $k = k_{j,N} \in \mathfrak{D}_{c,N}$. It is straightforward that (1) and (3′) in Condition 3.1 hold with $\mu^N = \tilde{\mu}^N_{\text{le}}$. The choice of Poisson distributions at sites $\mathfrak{D}_{c,N}$ allows for some explicit computation.

Fix any $c_0 \in (0, R(1/\lambda_{\max}))$, we now argue that $H(\tilde{\mu}^N_{\text{le}}|\mathscr{R}^N_{c_0}) = O(N)$. It suffices to check that $H(\mathcal{P}'_{aN}|\mathcal{P}_\varphi) = O(N)$ for any fixed $a \geq 0$ and $\varphi \in (0,1)$. To see this, let $f_N(n) = \mathcal{P}'_{aN}(X = n)$ and $f(n) = \mathcal{P}_\varphi(X = n)$ where $X(n) = n$. Then $H(\mathcal{P}'_{aN}|\mathcal{P}_\varphi) = E_{\mathcal{P}'_{aN}}[\ln f_N(X)] - E_{\mathcal{P}'_{aN}}[\ln f(X)]$. The term $E_{\mathcal{P}'_{aN}}[\ln f_N]$ is computed as $aN\ln(aN) - aN - E_{\mathcal{P}'_{aN}}[\ln X!]$. By Stirling's formula, $n! \geq \sqrt{2\pi n}e^{-n}n^n \geq e^{-n}n^n$, and Jensen's inequality, we have $E_{\mathcal{P}'_{aN}}[\ln X!] \geq E_{\mathcal{P}'_{aN}}[X\ln X - X] \geq aN\ln(aN) - aN$. Therefore, we have $E_{\mathcal{P}'_{aN}}[\ln f_N]$ is $O(N)$. For the term $E_{\mathcal{P}'_{aN}}[\ln f]$, we may write it as $aN\ln\varphi - \ln Z(\varphi) - E_{\mathcal{P}'_{aN}}[\ln g(X)!]$. Note that $g(X) \leq 1$, so that $E_{\mathcal{P}'_{aN}}[\ln f] = O(N)$. Hence, $H(\tilde{\mu}^N_{\text{le}}|\mathscr{R}^N_{c_0}) = O(N)$.

We now summarize the above calculations.

**Lemma 3.4.** *Suppose $g$ is bounded. Then $\mu^N = \tilde{\mu}^N_{le}$ satisfies Condition 3.1*

## 4. Results

4.1. **Hydrodynamic limits.** On $\mathbb{T}_N$, we will observe the evolution of the zero-range process speeded up by $N^2$, in diffusive scale. Denote the process $\eta_t := \xi_{N^2t}$, generated by $N^2L_N$ (cf. (2.1)), for times $0 \leq t \leq T$. Here, $T > 0$ refers to a fixed time horizon. We will access the space-time structure of the process through the scaled mass empirical measure:

$$\pi^N_t(dx) := \frac{1}{N}\sum_{k\in\mathbb{T}_N\setminus\mathfrak{D}_{s,N}} \eta_t(k)\delta_{k/N}(dx).$$

Throughout, we will view each $\pi^N_t$ as a member of $\mathcal{M}$, the space of finite nonnegative measures on $\mathbb{T}\setminus\mathfrak{D}_s$. We will place a metric $d(\cdot,\cdot)$ on $\mathcal{M}$ which realizes the vague convergence on $\mathbb{T}\setminus\mathfrak{D}_s$, (see Section 6 for a definitive choice). Here, the trajectories $\{\pi^N_t : 0 \leq t \leq T\}$ are elements of the Skorokhod space $D([0,T],\mathcal{M})$, endowed with the associated Skorokhod topology.

In the following, the process measure and associated expectation governing $\eta_\cdot$ starting from $\mu$ will be denoted by $\mathbb{P}_\mu$ and $\mathbb{E}_\mu$. When the process starts from $\{\mu^N\}_{N\in\mathbb{N}}$, in the class satisfying Condition 3.1, we will denote by $\mathbb{P}_N := \mathbb{P}_{\mu^N}$ and $\mathbb{E}_N := \mathbb{E}_{\mu^N}$, the associated process measure and expectation.

Suppose that $\{\mu^N\}$ satisfies Condition 3.1. Consequently, $\mu^N$ has macroscopic profile

$$\pi(dx) = \rho_0(x)dx + \sum_{j\in J_c}\mathfrak{m}_{0,j}\,\delta_{x_j}(dx)$$

on $\mathbb{T}\setminus\mathfrak{D}_s$ and we have $H(\mu^N|\mathscr{R}^N_{c_0}) = O(N)$ for some $c_0 \geq 0$.

**Theorem 4.1.** *Asssume $g$ is of $n^\alpha$ type. Then, for any $t > 0$, test function $G \in C(\mathbb{T})$, and $\delta > 0$, we have*

$$\lim_{N\to\infty}\mathbb{P}_N\Big[\Big|\langle G,\pi^N_t\rangle - \langle G,\pi_t\rangle\Big| > \delta\Big] = 0,$$

*where $\pi_t(dx) = \rho(t,x)dx + \sum_{j\in J_c}\mathfrak{m}_j(t)\,\delta_{x_j}(dx)$ is the unique weak solution to*

$$\begin{cases}\partial_t\pi_t = \partial_{xx}\Phi(\rho(t,x)), & x\in\mathbb{T}\setminus\mathfrak{D}_s,\ t\in(0,T),\\ \pi_t|_{t=0} = \pi, \quad \rho(t,x_j) = c_0, & t\in(0,T),\ j\in J_s,\\ \mathfrak{m}_j(t) = \left(\lambda_j\Phi(\rho(t,x_j))\right)^{1/\alpha}, & t\in(0,T),\ j\in J_c.\end{cases} \tag{4.1}$$

**Theorem 4.2.** *Assume $g$ is bounded. Then, for any $t > 0$, test function $G \in C(\mathbb{T})$, and $\delta > 0$, we have*

$$\lim_{N\to\infty} \mathbb{P}_N \Big[ \Big| \langle G, \pi_t^N \rangle - \langle G, \pi_t \rangle \Big| > \delta \Big] = 0,$$

*where $\pi_t(dx) = \rho(t,x)dx + \sum_{j\in J_c} \mathfrak{m}_j(t)\, \delta_{x_j}(dx)$ is the unique weak solution to*

$$\begin{cases} \partial_t \pi_t = \partial_{xx}\Phi(\rho(t,x)), & x \in \mathbb{T},\ t \in (0,T), \\ \pi_t|_{t=0} = \pi, \quad \Phi(\rho(t,x_j)) \le 1/\lambda_j, & t \in (0,T),\, j \in J_c, \\ \mathfrak{m}_j(t) = \mathfrak{m}_j(t) \mathbb{1}_{\rho(t,x_j)=R(1/\lambda_j)}, & t \in (0,T),\, j \in J_c. \end{cases} \tag{4.2}$$

We now define the weak solutions to the limit PDE (4.1) and (4.2).

**Definition 4.3.** *Let $f(t,x)$ and $g(t,x)$ be in $L^1_{loc}([0,T]\times D)$ where $D$ is a domain of $x$. We say $f$ is weakly differentialble with respect to $x \in D$ if for all $G(t,x) \in C_c^{0,1}([0,T]\times D)$ that*

$$\int_0^T \int_{\mathbb{T}} \partial_x G(t,x) f(t,x) dx dt = -\int_0^T \int_{\mathbb{T}} G(t,x) g(t,x) dx dt;$$

*The weak derivative will be denoted by $\partial_x f(t,x)$ and $\partial_x f(t,x) := g(t,x)$.*

**Definition 4.4.** *We say $\pi_t(dx) = \rho(t,x)dx + \sum_{j\in J_c} \mathfrak{m}_j(t)\, \delta_{x_j}(dx)$ is a weak solution to the system (4.1) if*

1. *$\rho(t,x)$ is in $L^2([0,T]\times\mathbb{T})$ and $\Phi(\rho(t,x))$ is weakly differentiable with respect to $x \in \mathbb{T}$ with $\partial_x\Phi(\rho(t,x)) \in L^2([0,T]\times\mathbb{T})$;*
2. *$\Phi(\rho(t,x_j)) = \Phi(c_0)$, for almost all $t \in (0,T)$ and all $x_j \in \mathfrak{D}_s$;*
3. *$\mathfrak{m}_j(t) = (\lambda_j \Phi(\rho(t,x_j)))^{1/\alpha}$ for almost all $t \in (0,T)$ and $j \in J_c$;*
4. *for all $G(t,x) \in C_c^\infty([0,T) \times (\mathbb{T}\setminus\mathfrak{D}_s))$*

$$\int_0^T \int_{\mathbb{T}} \partial_t G(t,x)\pi_t(dx)dt + \int_{\mathbb{T}} G(0,x)\pi(dx) + \int_0^T \int_{\mathbb{T}} \partial_{xx} G(t,x)\Phi(\rho(t,x))dxdt = 0. \tag{4.3}$$

**Remark 4.5.** *Notice that if $f(t,x)$ is weakly differentiable with respect to $x \in D$ as defined in Definition 4.3, then for a.e. $t \in [0,T]$, $f(t,\cdot)$ is absolutely continuous and $f(t,b)-f(t,a) = \int_a^b \partial_x f(t,x)dx$ for all connected $a, b \in D$. In particular, the evaluations of $\Phi(t,x)$ at $x = x_j$ in Definition 4.4 are well defined.*

*Moreover, since $\Phi$ is invertible, the evaluation $\Phi(\rho(t,x_j)) = \Phi(c_0)$ can be written as $\rho(t,x_j) = c_0$. Since $\Phi^{-1}$ is continuous, we also have that $\rho(t,x)$ is continuous in $x$.*

**Definition 4.6.** *We say $\pi_t(dx) = \rho(t,x)dx + \sum_{j\in J_c} \mathfrak{m}_j(t)\, \delta_{x_j}(dx)$ is a weak solution to the system (4.2) if*

1. *$\rho(t,x)$ is in $L^2([0,T]\times\mathbb{T})$ and $\Phi(\rho(t,x))$ is weakly differentiable with respect to $x \in \mathbb{T}$ with $\partial_x\Phi(\rho(t,x)) \in L^2([0,T]\times\mathbb{T})$;*
2. *$\Phi(\rho(t,x_j)) \le 1/\lambda_j$, for almost all $t \in (0,T)$ and all $x_j \in \mathfrak{D}_c$;*
3. *$\mathfrak{m}_j(t) = \mathfrak{m}_j(t)\mathbb{1}_{\Phi(\rho(t,x_j))=1/\lambda_j}$ for almost all $t \in (0,T)$ and $j \in J_c$;*
4. *for all $G(t,x) \in C_c^\infty([0,T) \times \mathbb{T})$*

$$\int_0^T \int_{\mathbb{T}} \partial_t G(t,x)\pi_t(dx)dt + \int_{\mathbb{T}} G(0,x)\pi(dx) + \int_0^T \int_{\mathbb{T}} \partial_{xx} G(t,x)\Phi(\rho(t,x))dxdt = 0.$$

To illustrate the relation between boundary conditions and effects on defect sites, we consider the case with only a single defect site.

**Example 4.7** (Effects of a single slow site)**.** Without loss of generality, we may assume the defect location is at 0 and the defect site has jump rate $(\lambda N^{\beta})^{-1}g(\xi(0))$.

Consider first $g$ of $n^{\alpha}$ type. By Theorem 4.1, the hydrodynamic limit $\pi_t$ is governed by the PDE (4.1). As the defect site is at $x = 0$, we have $\pi_t(dx) = \rho(t,x)dx$ when restricted to $(0,1)$. Then, the PDE (4.1) can be viewed as a one for $\rho(t,x)$ with different boundary conditions at $x = 0$ and $x = 1$ depending on the value of $\beta$. Precisely, we have the following.

(1) When $\beta < \alpha$, the defect site is invisible in the limit and (4.1) becomes

$$\begin{cases} \partial_t \rho(t,x) = \partial_{xx}\Phi(\rho(t,x)), & x \in \mathbb{T},\ t \in (0,T), \\ \rho(0,x) = \rho_0(x), \end{cases} \tag{4.4}$$

that is $\rho(t,x)$ satisfies periodic boundary conditions.

(2) When $\beta = \alpha$, $\pi_t(dx) = \rho(t,x)dx + \mathfrak{m}(t)\delta_0(dx)$ and $\mathfrak{m}(t) = (\lambda\Phi(\rho(t,0)))^{1/\alpha}$. As the total mass is conserved,

$$\mathfrak{m}_0 + \int_0^1 \rho_0(x)dx = \mathfrak{m}(t) + \int_0^1 \rho(t,x)dx, \quad \text{for all } t > 0.$$

Therefore, we have $\Phi(\rho(t,0)) = \lambda^{-1}\big[\mathfrak{m}_0 + \int_0^1(\rho_0(x) - \rho(t,x))dx\big]^{\alpha}$. Noticing that $x = 0$ and $x = 1$ coincide on $\mathbb{T}$, we obtain

$$\begin{cases} \partial_t \rho(t,x) = \partial_{xx}\Phi(\rho(t,x)), & x \in (0,1),\ t \in (0,T), \\ \rho(0,x) = \rho_0(x), \\ \Phi(\rho(t,0)) = \Phi(\rho(t,1)) = \lambda^{-1}\big[\mathfrak{m}_0 + \int_0^1 (\rho_0(x) - \rho(t,x))\, dx\big]^{\alpha}. \end{cases}$$

(3) when $\beta > \alpha$, it holds $\pi_t(dx) = \rho(t,x)dx$ on $\mathbb{T}$. As $\mathfrak{D} = \mathfrak{D}_s = \{0\}$, the PDE (4.1) is

$$\begin{cases} \partial_t \rho(t,x) = \partial_{xx}\Phi(\rho(t,x)), & x \in (0,1),\ t \in (0,T), \\ \rho(0,x) = \rho_0(x), \\ \Phi(\rho(t,0)) = \Phi(\rho(t,1)) = \Phi(c_0). \end{cases}$$

We now turn to case when $g$ is bounded. Notice that here we have $\beta \le 0$. When the defect is a fast site, that is $\beta < 0$ or $\beta = 0$ with $\lambda < 1$, by Theorem 4.2, the defect site is invisible macroscopically and the limit evolution is the usual nonlinear heat equation (4.4) with periodic boundary condition. When the defect is a slow site, that is $\beta = 0$ and $\lambda > 1$, the boundary condition moves between periodic and Dirichlet depending on whether there is mass at the defect. The macroscopic evolution is not closed if one considers only $\rho(t,x)$. Instead, we have $\pi_t(dx) = \rho(t,x)dx + \mathfrak{m}(t)\delta_0(dx)$ where

$$\begin{cases} \partial_t \pi_t = \partial_{xx}\Phi(\rho(t,x)), & & x \in \mathbb{T},\ t \in (0,T), \\ \mathfrak{m}(t) = \mathfrak{m}(t)\mathbb{1}_{\rho(t,0)=R(1/\lambda)}, & \rho(t,x) \le \Phi^{-1}(1/\lambda), & t \in (0,T), \\ \rho(0,x) = \rho_0(x), \quad \mathfrak{m}(0) = \mathfrak{m}_0. \end{cases}$$

We can also write $\mathfrak{m}(t)$ in term of 'mass conservation' as in part (2) above.

## 5. Stochastic differentials and proof outline

We analyze $\langle G, \pi_t^N \rangle$ by computing its stochastic differential in terms of certain martingales. Let $G$ be a smooth function with compact support on $[0,T] \times \mathbb{T} \setminus \mathfrak{D}_s$. Then, for $N$ large, $G_t(k_{j,N}/N) = 0$ for all $j \in J_s$. Let us write $G_t(x) := G(t,x)$.

We have

$$M_t^{N,G} = \langle G_t, \pi_t^N \rangle - \langle G_0, \pi_0^N \rangle - \int_0^t \Big\{ \langle \partial_s G_s, \pi_s^N \rangle + N^2 L_N \langle G_s, \pi_s^N \rangle \Big\} ds$$

is a mean zero martingale. Denote the discrete Laplacian $\Delta_N$ by

$$\Delta_N G\Big(\frac{k}{N}\Big) := N^2 \Big( G\Big(\frac{k+1}{N}\Big) + G\Big(\frac{k-1}{N}\Big) - 2G\Big(\frac{k}{N}\Big)\Big).$$

Then, for $N$ large, we compute

$$N^2 L_N \langle G, \pi_s^N \rangle = \frac{1}{N} \sum_{k \in \mathbb{T}_N} \Delta_N G_s\Big(\frac{k}{N}\Big) g_{k,N}(\eta_s(k)). \tag{5.1}$$

The quadratic variation of $M_t^{N,G}$ is given by

$$\langle M^{N,G} \rangle_t = \int_0^t \Big\{ N^2 L_N \Big( \langle G_s, \pi_s^N \rangle^2 \Big) - 2 \langle G_s, \pi_s^N \rangle N^2 L_N \langle G_s, \pi_s^N \rangle \Big\} ds$$

which by standard calculation equals

$$\int_0^t \sum_{k \in \mathbb{T}_N} g_{k,N}(\eta_s(k)) \Big\{ \Big( G_s(\frac{k+1}{N}) - G_s(\frac{k}{N}) \Big)^2 + \Big( G_s(\frac{k-1}{N}) - G_s(\frac{k}{N}) \Big)^2 \Big\} ds.$$

This variation may be bounded as follows.

**Lemma 5.1.** *For any test function $G(x) \in C_c^\infty(\mathbb{T} \setminus \mathfrak{D}_s)$, there is a constant $C$ independent of $N$ such that, for all $N$ large,*

$$\sup_{0 \le t \le T} \mathbb{E}_N \langle M^{N,G} \rangle_t \le C N^{-1}.$$

*Proof.* Since $G$ is smooth, we obtain, for $N$ large

$$\mathbb{E}_N \langle M^{N,G} \rangle_t \le 2(\|\partial_x G\|_\infty)^2 N^{-1} \mathbb{E}_N \Big[ \int_0^t \frac{1}{N} \sum_{k \in \mathbb{T}_N \setminus \mathfrak{D}_{s,N}} g_{k,N}(\eta_s(k)) ds \Big].$$

Note that $g(\cdot)$ grows at most linearly. Then, the lemma follows from the next Lemma 5.2. □

**Lemma 5.2.** *We have the following:*

(1) *The expectation of total mass at all but super-critical sites is uniformly bounded:*

$$\sup_{N \in \mathbb{N}} \sup_{t \ge 0} \mathbb{E}_N \Big[ \frac{1}{N} \sum_{k \in \mathbb{T}_N \setminus \mathfrak{D}_{s,N}} \eta_t(k) \Big] < \infty; \tag{5.2}$$

(2) *The expected particle number at each regular site $k \neq \mathfrak{D}_{s,N}$ is uniformly bounded:*

$$\sup_{N \in \mathbb{N}} \sup_{k \notin \mathfrak{D}_N} \sup_{t \ge 0} \mathbb{E}_N \Big[ \eta_t(k) \Big] < \infty; \tag{5.3}$$

(3) *The expectation of weighted jumping rate $N^{-1} g_{k,N}$ vanishes uniformly as $N \to \infty$:*

$$\lim_{N \to \infty} \sup_{t \ge 0} \sup_{k \in \mathbb{T}_N} \mathbb{E}_N \Big[ \frac{1}{N} g_{k,N}(\eta_t(k)) \Big] = 0. \tag{5.4}$$

*Proof.* The total mass estimate (5.2) follow directly from the initial bound of the entropy $H(\mu^N|\mathscr{R}^N_{c_0})$ and the entropy inequality. Similar argument also proves (5.4) for $k \notin \mathfrak{D}_{b,N}$ when $g$ is of $n^\alpha$ type. When $g$ is bounded, (5.4) holds trivially as long as $k$ is not in $\mathfrak{D}_{b,N}$ with $\beta_j < 0$.

It remains to check (5.3) for $k \notin \mathfrak{D}_N$ as well as (5.4) for defect sites $k_{j,N}$ with $\beta_j < 0$. In fact, they follow directly from the attractiveness result $\mu^N_t \leq \mathscr{R}^N_{c'}$ and $\mu^N_t \leq \overline{\kappa}^N_{c'}$ for $g$ of $n^\alpha$ type and bounded type respectively (see Section 3.2). □

5.1. **Proof outline of Theorems 4.1 and 4.2.** We sketch the proofs of Theorems 4.1 and 4.2. Let $Q^N$ be the probability measure on the trajectory space $D([0,T],\mathcal{M})$ governing $\pi^N_\cdot$ when the process starts from $\mu^N$. By Lemma 6.1, the family of measures $\{Q^N\}_{N\in\mathbb{N}}$ is tight with respect to the uniform topology, stronger than the Skorokhod topology. Let now $Q$ be any limit measure. We will show that $Q$ is supported on a class of weak solutions to the nonlinear PDE (4.1).

*Step 1.* Let $G(t,x)$ be a smooth function with compact support in $[0,T)\times(\mathbb{T}\setminus\mathfrak{D}_s)$. Recall the martingale $M^{N,G}_t$ and its quadratic variation $\langle M^{N,G}\rangle_t$ in the last section. By Lemma 5.1, we have $\mathbb{E}_N\big(M^{N,G}_T\big)^2 = \mathbb{E}_N\langle M^{N,G}\rangle_T$ vanishes as $N\to\infty$. By Doob's inequality, for each $\delta>0$,

$$\begin{aligned}&\mathbb{P}_N\Big[\big|\big\langle G_T,\pi^N_T\big\rangle - \big\langle G_0,\pi^N_0\big\rangle - \int_0^T \big(\big\langle \partial_s G_s,\pi^N_s\big\rangle + N^2L_N\big\langle G_s,\pi^N_s\big\rangle\big)ds\big| > \delta\Big]\\ &\leq \mathbb{P}_N\Big[\sup_{0\leq t\leq T}\big|M^{N,G}_t\big|>\delta\Big] \leq \frac{4}{\delta^2}\mathbb{E}_N\big\langle M^{N,G}\big\rangle_T \;\to\; 0 \;\text{ as } N\to\infty.\end{aligned}$$

Note that $G_T(x)=0$ and recall the evaluation of $N^2L_N\left\langle G_s,\pi^N_s\right\rangle$ in (5.1). Then,

$$\lim_{N\to\infty}\mathbb{P}_N\Big[\Big|\left\langle G_0,\pi^N_0\right\rangle + \int_0^T\Big\{\left\langle\partial_s G_s,\pi^N_s\right\rangle + \frac{1}{N}\sum_{k\in\mathbb{T}_N}\Delta_N G_s\big(\frac{k}{N}\big)g_{k,N}(\eta_s(k))\Big\}ds\Big|>\delta\Big]=0.$$

Let $\mathfrak{D}^\varepsilon = \cup_{j\in J}(x_j-\varepsilon,x_j+\varepsilon)$ and $F_\varepsilon(s,x)=\mathbb{1}_{\mathbb{T}\setminus\mathfrak{D}^\varepsilon}(x)\partial_{ss}G(s,x)$. By Lemma 5.2, we may replace $\Delta_N G_s(\cdot)$ by $\partial_{xx}G_s(\cdot)$ and then by $F_\varepsilon(s,\cdot)$ to obtain

$$\lim_{\varepsilon\to0}\lim_{N\to\infty}\mathbb{P}_N\Big[\Big|\left\langle G_0,\pi^N_0\right\rangle + \int_0^T\Big\{\left\langle\partial_s G_s,\pi^N_s\right\rangle + \frac{1}{N}\sum_{k\in\mathbb{T}_N}F_\varepsilon\big(\frac{k}{N}\big)g_{k,N}(\eta_s(k))\Big\}ds\Big|>\delta\Big]=0. \tag{5.5}$$

*Step 2.* We now replace the nonlinear term $g_{k,N}(\eta_s(k))$ by a function of the empirical density of particles. To be precise, let $\eta^l(x) = \dfrac{1}{2l+1}\sum_{|y-x|\leq l}\eta(y)$, that is the average density of particles in the box centered at $x$ with length $2l+1$. Therefore, using the Bulk Replacement Lemma (Lemma 8.4), we obtain from (5.5),

$$\begin{aligned}\lim_{\varepsilon\to0}\limsup_{\theta\to0}\limsup_{N\to\infty}\mathbb{P}_N\Big[\Big|\left\langle G_0,\pi^N_0\right\rangle + \int_0^T\Big\{\left\langle\partial_s G_s,\pi^N_s\right\rangle\\ + \frac{1}{N}\sum_{k\in\mathbb{T}_N}F_\varepsilon\big(s,\frac{k}{N}\big)\Phi\left(\eta^{\theta N}_t(k)\right)\Big\}ds\Big|>\delta\Big]=0.\end{aligned} \tag{5.6}$$

*Step 3.* For each $\theta>0$, take $\iota_\theta = (2\theta)^{-1}\mathbb{1}_{[-\theta,\theta]}$. The average density $\eta^{\theta N}_t(k)$ is written as a function of the empirical measure $\pi^N_t$

$$\eta^{\theta N}_t(k) = \frac{2\theta N}{2\theta N+1}\langle\iota_\theta(\cdot - k/N),\pi^N_t\rangle.$$

Since $\Phi$ is Lipschitz continuous and the total number of particles has expectation of order $N$ on the bulk (cf. Lemma 5.2), we may replace $\eta_t^{\theta N}(k)$ by $\langle \iota_\theta(\cdot - k/N), \pi_t^N \rangle$. Hence, we get from (5.6) in terms of the induced distribution $Q^N$ that

$$\lim_{\varepsilon\to 0}\limsup_{\theta\to 0}\limsup_{N\to\infty} Q^N\Big[\Big|\left\langle G_0, \pi_0^N\right\rangle + \int_0^T \Big\{ \left\langle \partial_s G_s, \pi_s^N\right\rangle \\ + \int_{\mathbb{T}} F_\varepsilon(s,x)\Phi\big(\langle \iota_\theta(\cdot - x), \pi_s^N\rangle\big)dx\Big\}ds\Big| > \delta\Big] = 0. \tag{5.7}$$

Notice that the discrete sum on $k$ is also replaced by the corresponding integral.

As the set of trajectories in (5.7) is open with respect to the Skorokhod topology, we obtain

$$\lim_{\varepsilon\to 0}\limsup_{\theta\to 0} Q\Big[\Big|\langle G_0, \pi_0\rangle + \int_0^T \Big\{ \langle \partial_s G_s, \pi_s\rangle \\ + \int_{\mathbb{T}} F_\varepsilon(s,x)\Phi\big(\langle \iota_\theta(\cdot - x), \pi_s\rangle\big)dx\Big\}ds\Big| > \delta\Big] = 0. \tag{5.8}$$

*Step 4.* We show in Lemma 7.1 that $Q$ is supported on trajectories

$$\pi_s(dx) = \rho(s,x)dx + \sum_{j\in J_c} \mathfrak{m}_j(s)\delta_{x_j}(dx).$$

Then, for $x \notin \mathfrak{D}^\varepsilon$ and $\theta < \varepsilon$, $\langle \iota_\theta(\cdot - x), \pi_s\rangle = (2\theta)^{-1}\int_{x-\theta}^{x+\theta}\rho(s,u)du$. Note that $\Phi$ is Lipschitz and $\rho$ is integrable on $[0,T]\times\mathbb{T}$. Hence, for all $\varepsilon$ small, we have, $Q$-almost surely

$$\lim_{\theta\to 0}\int_0^T\int_{\mathbb{T}} F_\varepsilon(s,x)\Phi\langle \iota_\theta(\cdot - x), \pi_s\rangle dxds = \int_0^T\int_{\mathbb{T}} F_\varepsilon(s,x)\Phi(\rho(s,x))dxds.$$

Since almost sure convergence impiies convergence in probability, we obtain from (5.8) that, for all $\delta > 0$

$$\lim_{\varepsilon\to 0} Q\Big[\Big|\langle G_0, \pi_0\rangle + \int_0^T \Big\{ \langle \partial_s G_s, \pi_s\rangle + \int_{\mathbb{T}} F_\varepsilon(s,x)\Phi(\rho(s,x))dx\Big\}ds\Big| > \delta\Big] = 0.$$

Taking $\varepsilon \to 0$ we may also replace $F_\varepsilon$ by $\partial_{xx}G$. As $\delta$ is arbitrary, we have

$$Q\Big[\langle G_0, \pi_0\rangle + \int_0^T \Big\{ \langle \partial_s G_s, \pi_s\rangle + \int_{\mathbb{T}} \partial_{xx}G_s\,(x)\,\Phi(\rho(s,x))dx\Big\}ds = 0\Big] = 1.$$

By Condition 3.1, the initial condition $\pi_0 = \pi$ holds. Thus, we conclude the limit measure $Q$ is concentrated on trajectories $\pi_\cdot$ that satisfies the weak formulation (4.3).

*Step 5.* That $\rho(s,x) \in L^2([0,T]\times\mathbb{T})$ follows from Lemma 7.1. The weak spacial differentiability of $\Phi(\rho(t,x))$ is addressed in Proposition 10.1. When $g$ is of $n^\alpha$ type, by Lemma 7.2 and Lemma 7.4, we obtain the boundary conditions $\mathfrak{m}_j(t) = (\lambda_j\Phi(\rho(t,x_j)))^{1/\alpha}$ for all $j \in J_c$ and $\Phi(\rho(t,x_j)) = \Phi(c_0)$ for all $j \in J_s$. When $g$ is bounded, by Lemma 7.5, it holds that $\Phi(\rho(t,x_j)) \le 1/\lambda_j$ and $\mathfrak{m}_j(t) = \mathfrak{m}_j(t)\mathbb{1}_{\Phi(\rho(t,x_j))=1/\lambda_j}$ for all $j \in J_c$. Therefore, $\pi_\cdot$ is a weak solution to (4.1) or (4.2) when $g$ is of $n^\alpha$ type or bounded respectively (cf. Definitions 4.4 and 4.6).

In section 11, we show that there is at most one weak solution $\pi_\cdot$ to (4.1) or (4.2). We conclude then that the sequence of $Q^N$ converges weakly to the Dirac measure on $\pi_\cdot$. Finally, as $Q^N$ converges to $Q$ with respect to the uniform topology, we have weak convergence of $\pi_t^N$ for each $0 \le t \le T$. That is, for all $G$ in $C_c(\mathbb{T}\setminus\mathfrak{D}_s)$, $\langle G, \pi_t^N\rangle$ weakly converges to the constant $\langle G, \pi_t\rangle$, and therefore convergence in probability as stated in Theorem 4.1 and

Theorem 4.2. (The extension to $G \in C(\mathbb{T})$ in Theorem 4.1 follows from (5.3) of Lemma 5.2.)

## 6. Tightness

Let $\mathcal{M}$ be the space of locally finite nonnegative measures on $\mathbb{T} \setminus \mathfrak{D}_s$. Let $C_c(\mathbb{T} \setminus \mathfrak{D}_s)$ be the space of continuous functions with compact support on $\mathbb{T} \setminus \mathfrak{D}_s$. Let $\{f_k\}_{k\in\mathbb{N}}$ be a countable dense set in $C_c(\mathbb{T} \setminus \mathfrak{D}_s)$ in the sense that for all $f \in C_c(\mathbb{T} \setminus \mathfrak{D}_s)$ there exists a subsequence $\{f_{n_k}\}$ such that $\operatorname{supp} f_{n_k} \subset \operatorname{supp} f$ for all $k$ and $f_{n_k}$ converges uniformly to $f$. Equipped with the distance

$$d(\mu,\nu) = \sum_{k=1}^{\infty} 2^{-k} \frac{\left|\int_0^1 f_k(d\mu - d\nu)\right|}{1 + \left|\int_0^1 f_k(d\mu - d\nu)\right|}.$$

the space $(\mathcal{M}, d(\cdot,\cdot))$ is a complete separable metric space. The metric $d(\cdot,\cdot)$ realizes the vague topology, that is, $\lim_{n\to\infty} d(\mu_n,\mu) = 0$ if and only if $\lim_{n\to\infty} \int f d\mu_n = \int f d\mu$ for all $f \in C_c(\mathbb{T} \setminus \mathfrak{D}_s)$.

6.1. **Tightness.** Recall the family of probability measures $\left\{Q^N\right\}_{N\in\mathbb{N}}$ on the trajectory space $D([0,T],\mathcal{M})$ in Section 5.1. In this section, we show $\left\{Q^N\right\}$ is tight with respect to the uniform topology, stronger than the Skorokhod topology on $D([0,T],\mathcal{M})$.

**Lemma 6.1.** $\left\{Q^N\right\}_{N\in\mathbb{N}}$ *is relatively compact with respect to the uniform topology. As a consequence, all limit points $Q$ are supported on trajectories $\{\pi_t\}_{t\in[0,T]}$ vaguely continuous on $\mathbb{T} \setminus \mathfrak{D}_s$, that is, for all $G \in C_c(\mathbb{T} \setminus \mathfrak{D}_s)$, the mapping $t \in [0,T] \mapsto \langle G, \pi_t\rangle$ is continuous.*

*Proof.* To deduce that $\{Q^N\}$ is relatively compact with respect to uniform topology, we show the following items (cf. Theorem 15.5 in [7]).

(1) For each $t \in [0,T]$, $\epsilon > 0$, there exists a compact set $K_{t,\epsilon} \subset \mathcal{M}$ such that

$$\sup_N Q^N \left[\pi_\cdot^N : \pi_t^N \notin K_{t,\epsilon}\right] \leq \epsilon. \tag{6.1}$$

(2) For every $\epsilon > 0$,

$$\limsup_{r\to 0} \limsup_{N\to\infty} Q^N \left[\pi_\cdot^N : \sup_{|t-s|<r} d(\pi_t^N, \pi_s^N) > \epsilon\right] = 0. \tag{6.2}$$

*Step 1.* We first consider (6.1). Notice that, for any $A > 0$, the set $\{\mu \in \mathcal{M} : \langle 1, \mu\rangle \leq A\}$ is compact in $\mathcal{M}$. Also, by (5.2), we have $\mathbb{E}_N\left[N^{-1}\sum_{k\notin\mathfrak{D}_{s,N}} \eta_t(k)\right] \leq C$ for some constant $C < \infty$ independent of $N$. As $Q^N\left[\langle 1, \pi_t^N\rangle > A\right] \leq A^{-1}\mathbb{E}_N\left[N^{-1}\sum_{k\notin\mathfrak{D}_{s,N}} \eta_t(k)\right]$, the first condition (6.1) is checked by taking $A$ large.

*Step 2.* To verify the second condition (6.2), it is enough to show a counterpart of the condition for the distributions of $\langle G, \pi_\cdot^N\rangle$ where $G$ is any smooth test function on $\mathbb{T} \setminus \mathfrak{D}_s$ (cf. p. 54, [17]). In other words, we need to show, for every $\epsilon > 0$,

$$\limsup_{r\to 0} \limsup_{N\to\infty} Q^N \left[\pi_\cdot^N : \sup_{|t-s|<r} \left|\langle G, \pi_t^N\rangle - \langle G, \pi_s^N\rangle\right| > \epsilon\right] = 0. \tag{6.3}$$

Note that $\left\langle G, \pi_t^N\right\rangle = \left\langle G, \pi_0^N\right\rangle + \int_0^t N^2 L_N \left\langle G, \pi_s^N\right\rangle ds + M_t^{N,G}$, then we only need to consider the oscillations of $\int_0^t N^2 L_N \left\langle G, \pi_s^N\right\rangle ds$ and $M_t^{N,G}$ respectively.

*Step 3.* For the oscillations of the martingale $M_t^{N,G}$, by $\big|M_t^{N,G} - M_s^{N,G}\big| \le \big|M_t^{N,G}\big| + \big|M_s^{N,G}\big|$, we have $\mathbb{P}_N\big[\sup_{|t-s|<r}\big|M_t^{N,G} - M_s^{N,G}\big| > \epsilon\big] \le 2\mathbb{P}_N\big[\sup_{0\le t\le T}\big|M_t^{N,G}\big| > \epsilon/2\big]$. Using Chebyshev and Doob's inequality, we further bound it by

$$\frac{8}{\epsilon^2}\mathbb{E}_N\Big[\Big(\sup_{0\le t\le T}\big|M_t^{N,G}\big|\Big)^2\Big] \le \frac{32}{\epsilon^2}\mathbb{E}_N\Big[\big(M_T^{N,G}\big)^2\Big] = \frac{32}{\epsilon^2}\mathbb{E}_N\langle M^{N,G}\rangle_T.$$

By Lemma 5.1, $\mathbb{E}_N\langle M^{N,G}\rangle_T = O(N^{-1})$. Then, we conclude

$$\lim_{r\to 0}\lim_{N\to\infty}\mathbb{P}_N\Big[\sup_{|t-s|<r}\Big|M_t^{N,G} - M_s^{N,G}\Big| > \epsilon\Big] = 0.$$

*Step 4.* To conclude (6.3), it suffices to show

$$\limsup_{r\to 0}\limsup_{N\to\infty} Q^N\Big[\sup_{|t-s|<r}\int_s^t \big|N^2 L_N\left\langle G, \pi_\tau^N\right\rangle\big| d\tau > \epsilon\Big] = 0.$$

The absolute value term is bounded above by $\|\Delta G\|_\infty \dfrac{1}{N}\sum_{k\notin\mathfrak{D}_{s,N}} g_{k,N}(\eta_\tau(k))$, cf. (5.1). Then, by Markov's inequality, it remains to show

$$\limsup_{r\to 0}\limsup_{N\to\infty}\mathbb{E}_N\Big[\sup_{|t-s|<r}\int_s^t \frac{1}{N}\sum_{k\in\mathbb{T}_N\setminus\mathfrak{D}_{s,N}} g_{k,N}(\eta_\tau(k))d\tau\Big] = 0.$$

By Lemma 5.2, $\mathbb{E}_N\big[\int_0^T N^{-1}g_{k,N}(\eta_t(k))dt\big]$ vanishes as $N\to\infty$ for defect sites $k\in\mathfrak{D}_{c,N}$ or $\mathfrak{D}_{b,N}$. Therefore, we may restrict the summation term in the previous display over $k\in\mathbb{T}_N\setminus\mathfrak{D}_N$. Note that for each such $k$ we have $g_{k,N}(\cdot) = g(\cdot)$. When $g$ is bounded, as $N^{-1}\sum_{k\notin\mathfrak{D}_N} g(\eta_\tau(k)) \le \|g\|_\infty$, the lemma is proved for this case.

The rest of this proof focuses on the case when $g$ is of $n^\alpha$ type. Since $g(\cdot)$ grows at most linearly, we are left to show

$$\lim_{r\to 0}\lim_{N\to\infty}\mathbb{E}_N\Big[\sup_{|t-s|<r}\Big|\int_s^t \frac{1}{N}\sum_{k\in\mathbb{T}_N\setminus\mathfrak{D}_N}\eta_\tau(k)d\tau\Big|\Big] = 0. \tag{6.4}$$

*Step 5.* To show (6.4). we introduce a trucation $\mathbb{1}_{\eta(k)\le A}$ with $A>0$. Notice that

$$\mathbb{E}_N\Big[\sup_{|t-s|<r}\Big|\int_s^t \frac{1}{N}\sum_{k\notin\mathfrak{D}_N}\eta_\tau(k)\mathbb{1}_{\eta_\tau(k)\le A}d\tau\Big|\Big] \le \mathbb{E}_N\Big[\sup_{|t-s|<r}\Big|\int_s^t A d\tau\Big|\Big] \le rA$$

which vanishes when taking $N\to\infty$, $r\to 0$, and $A\to\infty$ in order. It remains to show the error term with $\mathbb{1}_{\eta(k)>A}$ also vanishes in the limit. Note that the error term is estimated by

$$\begin{aligned}\mathbb{E}_N\Big[\sup_{|t-s|<r}\Big|\int_s^t \frac{1}{N}\sum_{k\notin\mathfrak{D}_N}\eta_\tau(k)\mathbb{1}_{\eta_\tau(k)>A}d\tau\Big|\Big] &\le \mathbb{E}_N\Big[\int_0^T \frac{1}{N}\sum_{k\notin\mathfrak{D}_N}\eta_\tau(k)\mathbb{1}_{\eta_\tau(k)>A}d\tau\Big]\\ &= \int_0^T \mathbb{E}_N\Big[\frac{1}{N}\sum_{k\notin\mathfrak{D}_N}\eta_\tau(k)\mathbb{1}_{\eta_\tau(k)>A}\Big]d\tau.\end{aligned}$$

By entropy inequality, for any $\tau\in[0,T]$ and $B>0$, the term $\mathbb{E}_N\Big[N^{-1}\sum\eta_\tau(k)\mathbb{1}_{\eta_\tau(k)>A}\Big]$ is further bounded above by

$$\frac{K_0}{B} + \frac{1}{BN}\ln E_{\mathscr{R}_c^N}\Big[\exp\Big\{B\sum_{k\notin\mathfrak{D}_N}\eta(k)\mathbb{1}_{\eta(k)>A}\Big\}\Big]. \tag{6.5}$$

Since, for $k \in \mathbb{T}_N \setminus \mathfrak{D}_N$, $\eta(k)$'s are i.i.d. with common distribution $\mathcal{P}_{\Phi(c)}$, (6.5) is equal to

$$\frac{K_0}{B} + \frac{N-n_0}{BN} \ln E_{\mathcal{P}_{\Phi(c)}}\left[e^{BX\mathbb{1}_{X>A}}\right] \leq \frac{K_0}{B} + B^{-1} \ln E_{\mathcal{P}_{\Phi(c)}}\left[e^{BX\mathbb{1}_{X>A}}\right].$$

Recall, $n_0$ is the number of defect sites; here, the distribution of $X$ is $\mathcal{P}_{\Phi(c)}$. As $E_{\mathcal{P}_{\Phi(c)}}[e^{BX}] < \infty$ for all $B > 0$. we have $B^{-1} \ln E_{\mathcal{P}_{\Phi(c)}}\left[e^{BX\mathbb{1}_{X>A}}\right] \to 0$ when taking $A \to \infty$ and then $B \to \infty$. Hence, we obtain

$$\lim_{A\to\infty} \lim_{r\to 0} \lim_{N\to\infty} \mathbb{E}_N\Big[\sup_{|t-s|<r}\Big|\int_s^t \frac{1}{N}\sum_{k\notin\mathfrak{D}_N} \eta_\tau(k)\mathbb{1}_{\eta_\tau(k)>A}d\tau\Big|\Big] = 0$$

which completes the proof. □

## 7. Properties of limit measures

By Lemma 6.1, the sequence $\{Q^N\}$ is relatively compact with respect to the uniform topology. Consider any convergent subsequence of $Q^N$ and relabel so that $Q^N \Rightarrow Q$. We now consider absolute continuity and boundary behaviors at defect sites for trajectories under $Q$.

### 7.1. Absolute continuity.

We now address absolute continuity.

**Lemma 7.1.** *We have $Q$ is supported on trajectories whose constrain on $\mathbb{T}\setminus\mathfrak{D}_c$ is absolutely continuous and the density is in $L^2$:*

$$Q\Big[\pi_\cdot : \pi_t(dx) = \rho(t,x)dx + \sum_{j\in J_c} \mathfrak{m}_j(t)\delta_{x_j}(dx) \text{ with } \rho(t,\cdot) \in L^2(\mathbb{T}) \text{ for all } 0 \leq t \leq T\Big] = 1.$$

*Proof.* Let $D := \mathbb{T}\setminus\{\mathfrak{D}_s \cup \mathfrak{D}_c\}$ and $C_c^+(D)$ be the space of nonnegative continuous functions with compact support on $D$, equipped with topology of uniform convergence on compact sets. Take $\{G_n\}_{n\in\mathbb{N}}$ be a dense sequence of $C_c^+(D)$. The lemma will follow if we have, for some $c \geq 0$

$$Q\Big[\pi_\cdot : \langle G_n, \pi_t\rangle \leq \langle G_n, c\rangle \text{ for all } 0 \leq t \leq T \text{ and } n \in \mathbb{N}\Big] = 1.$$

To this end, recall $\kappa_{c'}^N$ defined in Condition 3.1 and let $\nu^N$ denote $\mathscr{R}_{c'}^N$ when $g$ is of $n^\alpha$ type and $\kappa_{c'}^N$ when $g$ is bounded. By the product structure of $\nu^N$ and Chebyshev inequality, for each $\delta > 0$, we have $\nu^N[|\langle G_n, \pi^N\rangle - \langle G_n, c'\rangle| > \delta] \to 0$ as $N \to \infty$.

Fix $\varepsilon > 0$. By attractiveness (cf. Section 3.2), $Q^N\left[\langle G_n, \pi_t^N\rangle \leq \langle G_n, c'\rangle + \varepsilon\right]$ is bounded from below by $\nu^N\left[\langle G_n, \pi^N\rangle \leq \langle G_n, c'\rangle + \varepsilon\right]$. Then, we have, for all $t \geq 0$ and $\varepsilon > 0$

$$\lim_{N\to\infty} Q^N\left[\langle G_n, \pi_t^N\rangle \leq \langle G_n, c'\rangle + \varepsilon\right] = 1.$$

As compactness of $\{Q^N\}$ was shown in the uniform topology in Lemma 6.1, the distribution of $\langle G_n, \pi_t^N\rangle$ under $Q^N$ converges weakly to $\langle G_n, \pi_t\rangle$ under $Q$. Hence, we have

$$Q\Big[\langle G_n, \pi_t\rangle \leq \langle G_n, \kappa\rangle + \varepsilon\Big] \geq \limsup_{N\to\infty} Q^N\Big[\langle G_n, \pi_t^N\rangle \leq \langle G_n, c'\rangle + \varepsilon\Big] = 1.$$

Since $Q$ is supported on vaguely continuous trajectories by Lemma 6.1, we obtain for all $\varepsilon > 0$, $Q\Big[\langle G_n, \pi_t\rangle \leq \langle G_n, \kappa\rangle + \varepsilon \text{ for all } 0 \leq t \leq T, n \in \mathbb{N}\Big] = 1$. Then, we conclude the lemma by taking $\varepsilon \to 0$. □

7.2. **Boundary behavior.** For each $\theta \in (0,1)$, let $\hat{\iota}_\theta : (0,\theta) \mapsto [0,1]$ be a compactly supported continuous function such that $\hat{\iota}_\theta(x) = 1$ for $x \in (\theta^2, \theta - \theta^2)$. We use $\|\hat{\iota}_\theta\|$ to denote the $L^1$ norm $\int_0^\theta \hat{\iota}_\theta(x)dx$ and $\|\hat{\iota}_\theta\|^{-1} := (\|\hat{\iota}_\theta\|)^{-1}$. Notice that $\left|\|\hat{\iota}_\theta\|^{-1} - \theta^{-1}\right|$ is bounded from above for all $\theta$ small since $\lim_{\theta\to 0}\left|\|\hat{\iota}_\theta\|^{-1} - \theta^{-1}\right| \le 2$.

We first describe behavior near 'super-slow' sites in the $n^\alpha$ setting.

**Lemma 7.2.** *Let $g$ be of $n^\alpha$ type. Then, for any $j \in J_s$, $\delta > 0$, and $G \in C_c^\infty(0,T)$, it holds*

$$\lim_{\theta\to 0} Q\Big[\Big|\int_0^T G(s)\big(\Phi\big(\rho_{x_j}^{+,\theta}(s)\big) - \Phi(c_0)\big)ds\Big| > \delta\Big] = 0.$$

*where $\rho_{x_j}^{+,\theta}(s) := \|\hat{\iota}_\theta\|^{-1}\langle\hat{\iota}_\theta(\cdot - x_j), \rho\rangle = \|\hat{\iota}_\theta\|^{-1}\int_{\mathbb{T}}\hat{\iota}_\theta(x - x_j)\rho(s,x)dx$.*

*Proof.* We split the proof into steps.

*Step 1.* We first show that

$$\lim_{N\to\infty}\sup_{s\ge 0}\mathbb{E}_N\left[\left|N^{-\beta_j}g(\eta_s(k_{j,N})) - N^{-\beta_j}(\eta_s(k_{j,N}))^\alpha\right|\right] = 0. \tag{7.1}$$

Note that $g(n) \sim n^\alpha$. For each $\epsilon > 0$, let $A = A(\epsilon)$ be such that $|n^\alpha/g(n) - 1| < \epsilon$ for all $n > A$. Then, the expectation term in (7.1) is bounded above by $E_1 + E_2$ where $E_1$ and $E_2$ are the same expectation as in (7.1) with the integrand multiplied by indicators $\mathbb{1}_{\eta_s(k_{j,N})\le A}$ and $\mathbb{1}_{\eta_s(k_{j,N})>A}$ respectively. For each $A$, the term $E_1$ vanishes as $N \to \infty$. By the definition of $A$ and Lemma 5.2, the term $E_2$ is further bounded above by $\epsilon\sup_{s\ge 0}\mathbb{E}_N\left[N^{-\beta_j}g(\eta_s(k_{j,N}))\right] = O(\epsilon)$. Letting $\epsilon \to 0$, we conclude (7.1).

*Step 2.* We now argue that $N^{-\beta_j}(\eta_s(k_{j,N}))^\alpha$ may be replaced by $N^{-\beta_j}(\eta_0(k_{j,N}))^\alpha$. Let $r_{N,s} = N^{-\beta_j/\alpha}(\eta_s(k_{j,N}) - \eta_0(k_{j,N}))$. Let $\tau > 0$ be a constant such that there are no other defects within a $\tau$-neighborhood of $x_j$. Take a test function $F : \mathbb{T} \mapsto [0,1]$ such that $F$ has compact support in $(x_j - \tau, x_j + \tau)$ and $F(x) = 1$ for $|x - x_j| \le \tau/2$. Notice that

$$|r_{N,s}| \le N^{1-\beta_j/\alpha}\left|\langle F, \pi_s^N\rangle - \langle F, \pi_0^N\rangle\right| + N^{-\beta_j/\alpha}\sum_k^\circ(\eta_s(k) + \eta_0(k)) := I_1 + I_2.$$

where the $\sum_k^\circ$ term is summation over $k \in \mathbb{T}_N$ such that $|k/N - x_j| \le \tau$ and $k \ne k_{j,N}$.

As $\beta_j > \alpha$, Lemma 5.2 asserts that $\sup_{s\ge 0}\mathbb{E}_N[I_2]$ vanishes as $N \to \infty$.

To show $\mathbb{E}_N[I_1]$ vanishes, notice that $\left\langle F, \pi_s^N\right\rangle - \left\langle F, \pi_0^N\right\rangle = \int_0^s N^2 L_N\left\langle F, \pi_t^N\right\rangle dt + M_s^{N,F}$. Recall the generator computation (5.1) and then we may use Lemma 5.2 to obtain that $\sup_N\sup_{t\ge 0}\mathbb{E}_N\left[N^2 L_N\left\langle F, \pi_t^N\right\rangle\right]$ is finite. Also, by Lemma 5.1, the martingale term vanishes as its variance $\sup_{0\le s\le T}\mathbb{E}_N\langle M^{N,G}\rangle_s = o(1)$.

Hence, $\mathbb{E}_N[|r_{N,s}|]$, vanishes as $N \to \infty$ uniformly for all $s \in [0,T]$. As $\alpha \in (0,1]$, by the elementary inequality $(|x| + |y|)^\alpha \le |x|^\alpha + |y|^\alpha$, we have $N^{-\beta_j}|(\eta_s(k_{j,N}))^\alpha - (\eta_0(k_{j,N}))^\alpha| \le |r_{N,s}|^\alpha$. Therefore, we conclude

$$\lim_{N\to\infty}\sup_{0\le s\le T}\mathbb{E}_N\left[N^{-\beta_j}\left|(\eta_s(k_{j,N}))^\alpha - (\eta_0(k_{j,N}))^\alpha\right|\right] = 0. \tag{7.2}$$

In particular, it follows from (7.1) and (7.2) that, for any $\delta > 0$,

$$\lim_{N\to\infty}\mathbb{P}_N\left[\left|\int_0^T G(s)\big(N^{-\beta_j}g(\eta_s(k_{j,N})) - N^{-\beta_j}(\eta_0(k_{j,N}))^\alpha\big)ds\right| > \delta\right] = 0. \tag{7.3}$$

*Step 3.* In this step, we show that $N^{-\beta_j}(\eta_0(k_{j,N}))^\alpha$ may be replaced by $\varphi = \lambda_j\Phi(c_9)$. Precisely, as $\eta_0$ has distribution $\mu^N$, we show that

$$\lim_{N\to\infty}\mu^N\left[\left|N^{-\beta_j/\alpha}\xi(k_{j,N}) - \varphi^{1/\alpha}\right| > \delta\right] = 0. \tag{7.4}$$

Indeed, let $\mathcal{U}_N := \{|N^{-\beta_j/\alpha}\xi(k_{j,N}) - \varphi^{1/\alpha}| \geq \delta\}$. Fix $r_0 \in (1, \beta_j/\alpha)$. By the entropy inequality,

$$\mu^N[\mathcal{U}_N] = E_{\mu^N}[\mathbb{1}_{\mathcal{U}_N}] \leq N^{-r_0} H(\mu^N|\mathscr{R}_c^N) + N^{-r_0} \ln E_{\mathscr{R}_c^N}\big[e^{N^{r_0}\mathbb{1}_{\mathcal{U}_N}}\big].$$

Since $E_{\mathscr{R}_c^N}\big[e^{N^{r_0}\mathbb{1}_{\mathcal{U}_N}}\big] = (1-\mathscr{R}_c^N[\mathcal{U}_N]) + e^{N^{r_0}}\mathscr{R}_c^N[\mathcal{U}_N]$, we have

$$\lim_{N\to\infty} \mu^N[\mathcal{U}_N] \leq \max\big\{0, \lim_{N\to\infty} N^{-r_0}\ln\big(e^{N^{r_0}}\mathscr{R}_c^N[\mathcal{U}_N]\big)\big\}.$$

By the following Lemma 7.3, we have $\lim_{N\to\infty} N^{-r_0}\ln\mathscr{R}_c^N[\mathcal{U}_N] = -\infty$, and therefore, (7.4) holds.

*Step 4.* For each $\theta > 0$, let $\eta_s^{\theta N,+}(k) = (\theta N)^{-1}\sum_{0<j-k\leq\theta N}\eta_s(k)$. By Lemma 9.2, we may further replace $N^{-\beta_j}g(\eta_s(k_{j,N}))$ in (7.3) by $\lambda_j\Phi(\eta_s^{\theta N,+}(k_{j,N}))$ and obtain

$$\lim_{\theta\to 0}\lim_{N\to\infty}\mathbb{P}_N\Big[\Big|\int_0^T G(s)\big(\Phi(\eta_s^{\theta N,+}(k_{j,N})) - \Phi(c)\big)ds\Big| > \delta\Big] = 0. \tag{7.5}$$

Notice that $Q\big[\big|\int_0^T G(s)\big(\Phi\big(\rho_{x_j}^{+,\theta}(s)\big) - \Phi(c_0)\big)ds\big| > \delta\big]$ is bounded above by

$$\liminf_{N\to\infty} Q^N\Big[\Big|\int_0^T G(s)\big(\Phi\big(\|\hat{\iota}_\theta\|^{-1}\langle\hat{\iota}_\theta(\cdot - x_j), \pi_s^N\rangle\big) - \Phi(c_0)\big)ds\Big| > \delta\Big].$$

To conclude the lemma from (7.5), it remains to show that

$$\lim_{\theta\to 0}\lim_{N\to\infty}\mathbb{P}_N\Big[\int_0^T \big|\Phi(\eta_s^{\theta N,+}(k_{j,N})) - \Phi\big(\|\hat{\iota}_\theta\|^{-1}\langle\hat{\iota}_\theta(\cdot - x_j), \pi_s^N\rangle\big|ds > \delta\Big] = 0.$$

Since $\Phi(\cdot)$ is Lipschitz and $\big|\eta_s^{\theta N,+}(k_{j,N}) - \|\hat{\iota}_\theta\|^{-1}\langle\hat{\iota}_\theta(\cdot - x_j), \pi_s^N\rangle\big|$ is bounded above by

$$\big|\|\hat{\iota}_\theta\|^{-1} = \theta^{-1}\big|N^{-1}\sum_{0<k/N-x_j\leq\theta}\eta_s(k) + (\theta N)^{-1}\sum_{(k/N-x_j)\in(0,\theta^2)\cup(\theta-\theta^2,\theta]}\eta_s(k),$$

the claim follows from $\big|\|\hat{\iota}_\theta\|^{-1} = \theta^{-1}\big|$ being uniormly bounded for small $\theta$ and (5.3). □

**Lemma 7.3.** *Let $g$ be of $n^\alpha$ type. Let $X_N$ be distributed according to $\mathcal{P}_{N^\beta\varphi}$ for some $\beta > \alpha$ and $\varphi \geq 0$. Then, for any $r_0 < \beta/\alpha$ and $\delta > 0$*

$$\lim_{N\to\infty} N^{-r_0}\ln P\big[|N^{-\beta/\alpha}X_N - \varphi^{1/\alpha}| \geq \delta\big] = -\infty.$$

*Proof.* Let $\phi = \varphi^{1/\alpha}$. To prove the lemma, it suffices to show that

$$\lim_{N\to\infty} N^{-r_0}\ln P\big[X_N \geq (\phi+\delta)N^{\beta/\alpha}\big] = \lim_{N\to\infty} N^{-r_0}\ln P\big[X_N \leq (\phi-\delta)N^{\beta/\alpha}\big] = -\infty.$$

To this end, we notice that, for any $x > 0$,

$$\begin{aligned}\ln P\big[X_N \geq (\phi+\delta)N^{\beta/\alpha}\big] &\leq \ln\big(e^{-(\phi+\delta)N^{\beta/\alpha}x}E[e^{xX_N}]\big)\\ &= -(\phi+\delta)N^{\beta/\alpha}x + \ln\frac{Z(e^x N^\beta\varphi)}{Z(N^\beta\varphi)}.\end{aligned}$$

By Lemma 2.5, $\ln\dfrac{Z(e^x N^\beta\varphi)}{Z(N^\beta\varphi)} \sim \alpha\phi N^{\beta/\alpha}(e^{x/\alpha}-1)$. Then, for

$$K_{\delta,\phi,\alpha} := \max_{x>0}\big\{(\phi+\delta)x - \alpha\phi(e^{x/\alpha}-1)\big\}$$

we have

$$\limsup_{N\to\infty} N^{-\beta/\alpha}\ln P\big[X \geq (\phi+\delta)N^{\beta/\alpha}\big] \leq -K_{\delta,\phi,\alpha} < 0.$$

Since $r_0 < \beta/\alpha$, we obtain

$$\lim_{N\to\infty} N^{-r_0} \ln P\big[X_N \geq (\phi+\delta)N^{\beta/\alpha}\big] = -\infty. \tag{7.6}$$

On the other hand, for all $x > 0$, we also have

$$\ln P\big[X_N \leq (\phi-\delta)N^{\beta/\alpha}\big] \leq \ln\Big(e^{(\phi-\delta)N^{\beta/\alpha}x} E[e^{-xX_N}]\Big).$$

By a similar argument employed to prove (7.6), we have

$$\lim_{N\to\infty} N^{-r_0} \ln P\big[X_N \leq (\phi-\delta)N^{\beta/\alpha}\big] = -\infty.$$

The lemma is now proved. □

We now consider the behavior near 'critical' slow sites in the $n^\alpha$ setting.

**Lemma 7.4.** *Let $g$ be of $n^\alpha$ type. Then, for any $j \in J_c$, $\delta > 0$, and $G \in C_c^\infty(0,T)$, we have*

$$\lim_{\theta\to 0} Q\Big[\Big|\int_0^T G(s)\Big(\lambda_j \Phi\big(\rho_{x_j}^{+,\theta}(s)\big) - (\mathfrak{m}_j(t))^\alpha\Big) ds\Big| > \delta\Big] = 0,$$

*where $\rho_{x_j}^{+,\theta}(s) = \|\hat{\iota}_\theta\|^{-1}\langle \hat{\iota}_\theta(\cdot - x_j), \rho\rangle$.*

*Proof.* Fix $j \in J_c$. As $g(n) \sim n^\alpha$, arguments as in Step 1 of the proof of Lemma 7.2 shows that $N^{-\alpha}g(\eta_s(k_{j,N}))$ may be replaced by $N^{-\alpha}(\eta_s(k_{j,N}))^\alpha$. Hence, we have

$$\lim_{N\to\infty} \mathbb{P}_N\Big[\Big|\int_0^T G(s)\Big(N^{-\alpha}g(\eta_s(k_{j,N})) - (N^{-1}\eta_s(k_{j,N}))^\alpha\Big) ds\Big| > \delta\Big] = 0. \tag{7.7}$$

Let $\{F_\theta(x)\}$ be a sequence of nonnegative smooth functions such that $F_\theta$'s are supported on $(x_j - \theta, x_j + \theta)$, $\|F_\theta\|_\infty \leq 1$, and $F_\theta(x) = 1$ for $|x - x_j| \leq \theta/2$. Then,

$$|\langle F_\theta, \pi_s^N\rangle - N^{-1}\eta_s(k_{j,N})| \leq B_{\theta,N}(\eta_s)$$

where $B_{\theta,N}(\eta) = N^{-1}\sum_{k\notin\mathfrak{D}_N} \mathbb{1}_{(-\theta,\theta)}(k/N - x_j)\,\eta(k)$ for $\theta$ small and $N$ large. Note that

$$\mathbb{E}_N\big[\big|\langle F_\theta, \pi_s^N\rangle^\alpha - (N^{-1}\eta_s(k_{j,N}))^\alpha\big|\big] \leq \mathbb{E}_N\big[(B_{\theta,N})^\alpha\big] \leq \mathbb{E}_N\big[B_{\theta,N}\big]^\alpha$$

which vanishes as $N \to \infty$ and $\theta \to 0$ by Lemma 5.2. Therefore, we may replace $N^{-1}\eta_s(k_{j,N})$ in (7.7) to obtain

$$\lim_{\theta\to 0}\limsup_{N\to\infty} \mathbb{P}_N\Big[\Big|\int_0^T G(s)\Big(N^{-\alpha}g(\eta_s(k_{j,N})) - \langle F_\theta, \pi_s^N\rangle^\alpha\Big) ds\Big| > \delta\Big] = 0.$$

By Lemma 9.2, $N^{-\alpha}g(\eta_s(k_{j,N}))$ may be replaced by $\lambda_j\Phi(\eta_s^{\theta N,+}(k_{j,N}))$. Moreover, following Step 4 in the proof of Lemma 7.2 to replace $\eta_s^{\theta N,+}(k_{j,N})$ with $\rho_{x_j}^{+,\theta}(s)$, we obtain

$$\lim_{\theta\to 0} Q\Big[\Big|\int_0^T G(s)\Big(\lambda_j \Phi\big(\rho_{x_j}^{+,\theta}(s)\big) - \langle F_\theta, \pi_s\rangle^\alpha\Big) ds\Big| > \delta\Big] = 0.$$

Note that $\int_0^T G(s)\langle F_\theta, \pi_s\rangle^\alpha ds$ converges to $\int_0^T (\mathfrak{m}_j(s))^\alpha ds = \int_0^T (\pi_s(\{x_j\}))^\alpha ds$ almost surely with respect to $Q$. Hence, with these observations, the lemma is now proved. □

We now turn to the behavior near 'critical' slow sites in the $g$ bounded setting.

**Lemma 7.5.** *Let $g$ be bounded. Then, for any $j \in J_c$, $\delta > 0$, and $G \in C_c^\infty(0,T)$, we have*

$$\lim_{\theta\to 0} Q\Big[\int_0^T |G(s)|\big(\Phi\big(\rho_{x_j}^{+,\theta}(s)\big) - \phi_j\big)ds > \delta\Big] = 0 \tag{7.8}$$

*and*

$$\lim_{\theta\to 0} Q\Big[\Big|\int_0^T G(s)(\mathfrak{m}_j(t)\wedge 1)\big(\phi_j - \Phi\big(\rho_{x_j}^{+,\theta}(s)\big)\big)ds\Big| > \delta\Big] = 0. \tag{7.9}$$

*where $\rho_{x_j}^{+,\theta}(s) := \|\hat\iota_\theta\|^{-1}\langle\hat\iota_\theta(\cdot - x_j), \rho\rangle$, $\phi_j = \lambda_j^{-1}$, and $a\wedge b = \min\{a,b\}$.*

*Proof.* We first address (7.8). As $g(\cdot) \le 1$, it is trivial that, for all $N$

$$\mathbb{P}_N\Big[\int_0^T |G(s)|\left(\lambda_j^{-1}g(\eta_s(k_{j,N})) - \phi_j\right)ds > \delta\Big] = 0.$$

We now use the local 'replacement' Lemma 9.2 to replace $\lambda_j^{-1}g(\eta_s(k_{j,N}))$ by $\Phi(\eta_s^{\theta N,+}(k_{j,N}))$, to obtain

$$\lim_{\theta\to 0}\limsup_{N\to\infty}\mathbb{P}_N\Big[\int_0^T |G(s)|\left(\Phi(\eta_s^{\theta N,+}(k_{j,N})) - \phi_j\right)ds > \delta\Big] = 0.$$

Then, (7.8) follows by taking $N\to\infty$, cf Step 4 in Lemma 7.2.

To show (7.9), we take the same sequence $\{F_\theta(x)\}$ from the proof of Lemma 7.4. We now observe that

$$\lim_{\theta\to 0}\limsup_{N\to\infty}\mathbb{E}_N\Big[\int_0^T \left|\left(\langle F_\theta, \pi_s^N\rangle \wedge 1\right)\left(\phi_j - \lambda_j^{-1}g(\eta_s(k_{j,N}))\right)\right|ds\Big] = 0. \tag{7.10}$$

Indeed, as $\lim_{n\to\infty} g(n) = 1$, let $A = A(\varepsilon)$ be such that $|g(n) - 1| < \varepsilon$ for all $n \ge A$. Then, in the above display, on the one hand, the absolute value is bounded above by $\varepsilon$ when $\eta_s(k_{j,N}) \ge A$. On the other hand, $\langle F_\theta, \pi_s^N\rangle$ is less than $N^{-1}(A + \sum_k^{\circ,\theta}\eta_s(k))$ when $\eta_s(k_{j,N}) < A$. Here, the sum $\sum_k^{\circ,\theta}$ is over all $k$ such that $|k/N - x_j| \le \theta$ and $k \ne k_{j,N}$. In considering these cases, when $\eta_s(k_{j,N}) \le A$ and $\eta_s(k_{j,N}) \ge A$ respectively, the estimate (7.10) follows from 'particle numbers' Lemma 5.2.

Now, we observe that the local 'replacement' Lemma 9.2 remains effective when the test function is taken in form $G(s)\left(\langle F_\theta, \pi_s^N\rangle \wedge 1\right)$. Thus, we may replace $\lambda_j^{-1}g(\eta_s(k_{j,N}))$ by $\Phi(\eta_s^{\theta N,+}(k_{j,N}))$ and obtain

$$\lim_{\theta\to 0}\limsup_{N\to\infty}\mathbb{P}_N\Big[\Big|\int_0^T G(s)\left(\langle F_\theta, \pi_s^N\rangle \wedge 1\right)\left(\phi_j - \Phi(\eta_s^{\theta N,+}(k_{j,N}))\right)ds\Big| > \delta\Big] = 0.$$

Therefore, it holds $\lim_{\theta\to 0} Q\Big[\Big|\int_0^T G(s)\left(\langle F_\theta, \pi_s\rangle \wedge 1\right)\left(\phi_j - \Phi\big(\rho_{x_j}^{+,\theta}(s)\big)\right)ds\Big| > \delta\Big] = 0$. Since $\Phi$ is bounded and $\lim_{\theta\to 0}\int_0^T |(\langle F_\theta, \pi_s\rangle \wedge 1) - (\mathfrak{m}_j(s)\wedge 1)|\,ds = 0$ $Q$-almost surely, we conclude the proof of (7.9), finishing the proof. □

## 8. Local 1 and 2-block estimates of bulk sites

In this section we address the 1 and 2-block estimates for non-defect sites $\mathbb{T}_N \setminus \mathfrak{D}_N$. These estimates are obtained through a Rayleigh-type estimation of a variational eigenvalue expression derived from a Feynman-Kac bound.

8.1. **Local 1-block estimate.** We start from recalling the concept of spectral gap which is used to prove our local 1-block estimate. For $k \in \mathbb{T}_N$ and $l \geq 1$, define the set $\Lambda_{k,l} = \{k-l, k-l+1, \dots, k+l\} \subset \mathbb{T}_N$. Let $\Omega_{k,l} = \mathbb{N}_0^{\Lambda_{k,l}}$ be the state space of configurations restricted on sites $\Lambda_{k,l}$. Define the state space of configurations with exactly $j$ particles on the sites $\Lambda_{k,l}$:

$$\Omega_{k,l,j} = \Big\{\eta \in \Omega_{k,l} : \sum_{x \in \Lambda_{k,l}} \eta(x) = j\Big\}.$$

Consider the generator $L_{k,l}$ on $\Omega_{k,l}$ given by

$$L_{k,l} f(\eta) = \sum_{x,x+1 \in \Lambda_{k,l}} \Big\{ g(\eta(x)) \left[ f\left(\eta^{x,x+1}\right) - f(\eta) \right] + g(\eta(x)) \left[ f\left(\eta^{x+1,x}\right) - f(\eta) \right] \Big\}.$$

Recall the generator $L_N$ from (2.1). Notice that, for each $k, l$ such that $\Lambda_{k,l} \cap \mathfrak{D}_N = \emptyset$, the generator $L_{k,l}$ coincides with $L_N$ localized on $\Lambda_{k,l}$.

For any $\rho > 0$, let $\nu_\rho$ be the product measure on $\Omega = \mathbb{N}_0^{\mathbb{T}_N}$ with common marginal $\mathcal{P}_{\Phi(\rho)}$ on each site $k \in \mathbb{T}_N$, and let $\nu_{k,l}^\rho$ be its restriction to $\Omega_{k,l}$. Let $\nu_{k,l,j}$ be $\nu_{k,l}^\rho$ conditioned on total number of particles on $\Lambda_{k,l}$ being $j$. Notice that $\nu_{k,l,j}$ does not depend on $\rho$. It is well-known that both $\nu_{k,l}^\rho$ and $\nu_{k,l,j}$ are invariant measures with respect to the localized generator $L_{k,l}$ (cf. [1]). For $\kappa = \nu_{k,l}^\rho$ or $\nu_{k,l,j}$, the corresponding Dirichlet form is given by

$$E_\kappa \left[ f(-L_{k,l} f) \right] = \sum_{x,x+1 \in \Lambda_{k,l}} E_\kappa \left[ g(\eta(x)) \left( f\left(\eta^{x,x+1}\right) - f(\eta) \right)^2 \right]. \tag{8.1}$$

For $j \geq 1$, let $b_{l,j}$ be the spectral gap of $-L_{k,l}$ on $\Omega_{k,l,j}$ (cf. p. 374, [17]):

$$b_{l,j} := \inf_f \frac{E_{\nu_{l,j}}[f(-L_{k,l} f)]}{\mathrm{Var}_{\nu_{k,l,j}}(f)} \tag{8.2}$$

where the infimum is taken over all $L^2(\nu_{k,l,j})$ functions $f$ from $\Omega_{k,l,j}$ to $\mathbb{R}$. For all $l, j \geq 1$, as $\Omega_{k,l,j}$ is a finite space and the localized process is irreducible, we have $b_{l,j} > 0$. As a consequence, we have the following Poincaré inequality: for all $f \in L^2(\nu_{k,l,j})$

$$\mathrm{Var}_{\nu_{k,l,j}}(f) \leq C_{l,j} E_{\nu k,l,j} \left[ f(-L_{k,l} f) \right] \tag{8.3}$$

where $C_{l,j} := b_{l,j}^{-1} < \infty$ for $j \geq 1$ and $C_{l,0} = 0$. We remark that even though, for a large class of $g(\cdot)$'s, sharp estimates of $b_{l,j}$ are available in the literature, we will only need that $b_{l,j}$ is strictly positive for all $l, j \geq 1$.

We now prove the local 1-block estimate for regular sites:

**Lemma 8.1** (Local 1-block estimate)**.** *For any bounded function $G$ on $[0,T] \times \mathbb{T}$, we have*

$$\limsup_{l \to \infty} \limsup_{N \to \infty} \sup_{k,k'} \mathbb{E}_N \Big[ \Big| \int_0^T G(s, k'/N) \left( g(\eta_s(k')) - \Phi(\eta_s^l(k)) \right) ds \Big| \Big] = 0$$

*where the* sup *is taken over all $k$ and $k'$ such that $k' \in \Lambda_{k,l}$ and $\Lambda_{k,l} \cap \mathfrak{D}_N = \emptyset$.*

*Proof.* We separate the argument into 5 steps.

*Step 1.* We first introduce a cutoff of large densities. Let

$$V_{k,k',l}(s,\eta) := G(s,k'/N) \left( g(\eta(k')) - \Phi(\eta^l(k)) \right).$$

As $g(n) \leq g^* n$ and $\Phi(x) \leq g^* x$, we have

$$\mathbb{E}_N \Big[ \Big| \int_0^T V_{k,k',l}(s,\eta_s) \mathbb{1}_{\eta_s^l(k) > A} ds \Big| \Big] \leq g^* \|G\|_\infty \int_0^T \mathbb{E}_N \left[ \left( \eta_s(k') + \eta_s^l(k) \right) \mathbb{1}_{\eta_s^l(k) > A} \right] ds.$$

By attractiveness (cf. Section 3.2), in both $g(n) \sim n^\alpha$ and bounded settings (as $\Lambda_{k,l} \cap \mathfrak{D}_N = \emptyset$ so the marginals of $\kappa^N_{c'}$ and $\mathscr{R}^N_{c'}$ agree), the last expectation is bounded above by

$$\begin{aligned} E_{\mathscr{R}^N_{c'}}\left[\left(\eta(k') + \eta^l(k)\right)\mathbb{1}_{\eta^l(k)>A}\right] \le & A^{-1} E_{\nu_{c'}}\left[\left(\eta(k')\eta^l(k)\right) + \left(\eta^l(k)\right)^2\right] \\ \le & A^{-1} E_{\nu_{c'}}\left[\left(\eta(k')\right)^2 + 2\left(\eta^l(k)\right)^2\right]. \end{aligned}$$

Notice that $\left(\eta^l(k)\right)^2 \le (2l+1)^{-1}\sum_{j\in\Lambda_{k,l}}(\eta(j))^2$ and, under $\nu_{c'}$, $\{\eta(j)\}_{j\in\Lambda_{k,l}}$ has common distribution $\mathcal{P}_{\Phi(c')}$. we obtain $\mathbb{E}_N\left[\left|\int_0^T V_{k,k',l}(s,\eta_s)\mathbb{1}_{\eta^l_s(k)>A}ds\right|\right] \to 0$ as $N$, $l$, and $A$ approach $\infty$ in order.

Therefore, to prove the lemma, it will be enough to show, for all $A > 0$, that

$$\limsup_{l\to\infty}\limsup_{N\to\infty}\sup_{k,k'}\mathbb{E}_N\left[\left|\int_0^T V_{k,k',l,A}(s,\eta_s)ds\right|\right] = 0$$

where $V_{k,k',l,A}(s,\eta) := V_{k,k',l}(s,\eta)\mathbb{1}_{\{\eta^l(k)\le A\}}$.

*Step 2.* As $H(\mu^N|\mathscr{R}^N_{c_0}) \le CN$, it follows from the entropy inequality

$$\mathbb{E}_N\left[\left|\int_0^T V_{k,k',l,A}(s,\eta_s)ds\right|\right] \le \frac{C}{\gamma} + \frac{1}{\gamma N}\ln\mathbb{E}_{\mathscr{R}^N_{c_0}}\left[\exp\left\{\gamma N\left|\int_0^T V_{k,k',l,A}(s,\eta_s)ds\right|\right\}\right].$$

The absolute value in the right hand side of last inequality can be dropped by using $e^{|x|} \le e^x + e^{-x}$. By Feynman-Kac formula (cf. p.336, [17]),

$$\frac{1}{\gamma N}\ln\mathbb{E}_{\mathscr{R}^N_{c_0}}\left[\exp\left\{\gamma N\int_0^T V_{k,k',l,A}(s,\eta_s)ds\right\}\right] \le \frac{1}{\gamma N}\int_0^T \lambda_{N,l}(s)ds$$

where $\lambda_{N,l}(s)$ is the largest eigenvalue of $N^2 L_N + \gamma N V_{k,k',l,A}(s,\eta)$.

*Step 3.* Fix $s \in [0,T]$; we will omit the argument $s$ in $\lambda_{N,l}(s)$ to simplify notation, that is $\lambda_{N,l} = \lambda_{N,l}(s)$. Note the variational formula for $\lambda_{N,l}$:

$$(\gamma N)^{-1}\lambda_{N,l} = \sup_f\left\{E_{\mathscr{R}^N_{c_0}}\left[V_{k,k',l,A}f\right] - \gamma^{-1}NE_{\mathscr{R}^N_{c_0}}\left[\sqrt{f}(-L_N\sqrt{f})\right]\right\},$$

where the supremum is over all $f$ which are densities with respect to $\mathscr{R}^N_c$ (cf. [17], p. 377).

Let $f_{k,l} = E_{\mathscr{R}^N_{c_0}}\left[f|\Omega_{k,l}\right]$, be the conditional expectation of $f$ given the variables on $\Lambda_{k,l}$. For $\mu = \mathscr{R}^N_{c_0}$ let $\mu_{k,l}$ be $\mu$ restricted on $\Lambda_{k,l}$. Clearly, $\mu_{k,l} = \nu^{c_0}_{k,l}$. Since the Dirichlet form $E_{\mathscr{R}^N_{c_0}}\left[\sqrt{f}(-L_N\sqrt{f})\right]$ is convex, we have

$$(\gamma N)^{-1}\lambda_{N,l} \le \sup_{f_{k,l}}\left\{E_{\mu_{k,l}}\left[V_{k,k',l,A}f_{k,l}\right] - \gamma^{-1}NE_{\mu_{k,l}}\left[\sqrt{f_{k,l}}(-L_{k,l}\sqrt{f_{k,l}})\right]\right\}.$$

*Step 4.* We now decompose $f_{k,l}d\mu_{k,l}$ with respect to sets $\Omega_{k,l,j}$ of configurations with total particle number $j$ on $\Lambda_{k,l}$:

$$E_{\mu_{k,l}}\left[V_{k,k',l,A}f_{k,l}\right] = \sum_{j\ge 0} c_{k,l,j}(f)\int V_{k,k',l,A}f_{k,l,j}d\mu_{k,l,j}, \tag{8.4}$$

where $c_{k,l,j}(f) = \int_{\Omega_{k,l,j}} f_{k,l}d\mu_{k,l}$, and $f_{k,l,j} = c_{k,l,j}(f)^{-1}\mu_{k,l}\left(\Omega_{k,l,j}\right)f_{k,l}$. Here, $\sum_{j\ge 0} c_{k,l,j} = 1$ and $f_{k,l,j}$ is a density with respect to $\mu_{k,l,j}$.

Straightforwardly, on $\Omega_{k,l,j}$, we have

$$\frac{L_{k,l}\sqrt{f_{k,l}}}{\sqrt{f_{k,l}}} = \frac{L_{k,l}\sqrt{f_{k,l,j}}}{\sqrt{f_{k,l,j}}}.$$

Using (8.4), we write

$$E_{\mu_{k,l}}\left[\sqrt{f_{k,l}}(-L_{k,l}\sqrt{f_{k,l}})\right] = \sum_{j\geq 0} c_{k,l,j}(f) E_{\mu_{k,l,j}}\left[\sqrt{f_{k,l,j}}(-L_{k,l}\sqrt{f_{k,l,j}})\right].$$

Then, we get

$$(\gamma N)^{-1}\lambda_{N,l} \leq \sup_{0\leq j\leq A(2l+1)} \sup_f \left\{ E_{\mu_{k,l,j}}\left[V_{k,k',l,A}f\right] - \gamma^{-1} N E_{\mu_{k,l,j}}\left[\sqrt{f}(-L_{k,l}\sqrt{f})\right]\right\},$$

where the second supremum is on all densities $f$ with respect to $\mu_{k,l,j}$.

*Step 5.* Let

$$\widehat{V}_{k,k',l,A} = V_{k,k',l,A} - E_{\mu_{k,l,j}}\left[V_{k,k',l,A}\right].$$

Let $C_{l,A,G}$ be such that $\|\widehat{V}_{k,k',l,A}\|_\infty \leq C_{l,A,G}$. Recall $C_{l,j}$ the inverse spectral gap of $L_{k,l}$ (cf. (8.3)). We now use the Rayleigh expansion (cf. [17], pp. 375–376, Appendix 3, Theorem 1.1)

$$\begin{aligned} & E_{\mu_{k,l,j}}\left[\widehat{V}_{k,k',l,A}f\right] - \gamma^{-1} N E_{\mu_{k,l,j}}\left[\sqrt{f}(-L_{k,l}\sqrt{f})\right] \\ \leq & \frac{\gamma N^{-1}}{1-2C_{l,A,G}C_{l,j}\,\gamma N^{-1}} E_{\mu_{k,l,j}}\left[\widehat{V}_{k,k',l,A}(-L_{k,l})^{-1}\widehat{V}_{k,l,A}\right]. \end{aligned} \tag{8.5}$$

The spectral gap of $L_{k,l}$ also implies that $\|L_{k,l}^{-1}\|_2$, the $L^2(\mu_{k,l,j})$ norm of the operator $L_{k,l}^{-1}$ on mean zero functions, is less than or equal to $C_{l,j}$. Now, by Cauchy-Schwarz and the estimate of $\|L_{k,l}^{-1}\|_2$, we have

$$E_{\mu_{k,l,j}}\left[\widehat{V}_{k,k',l,A}(-L_{k,l})^{-1}\widehat{V}_{k,k',l,A}\right] \leq C_{l,j} E_{\mu_{k,l,j}}\left[\widehat{V}^2_{k,k',l,A}\right] \leq C_{l,j}C^2_{l,A,G}.$$

Accordingly, retracing our steps, noting (8.5), we have that $\mathbb{E}_N\left[\left|\int_0^T V_{k,k',l,A}(\eta_s)ds\right|\right]$ is less than or equal to

$$\frac{C_0}{\gamma} + \sup_{0\leq j\leq A(2l+1)} \frac{T\gamma N^{-1}C_{l,j}C^2_{l,A,G}}{1-2C(l,A,G)C_{l,j}\,\gamma N^{-1}} + T\sup_{0\leq j\leq A(2l+1)} E_{\mu_{k,l,j}}\left[V_{k,k',l,A}\right].$$

Taking $N\to\infty$, first sup term vanishes. Notice that the expression $\sup E_{\mu_{k,l,j}}\left[V_{k,k',l,A}\right]$ is independent of $N$ and vanishes as $l\to\infty$. In fact, as $\mu_{k,l,j}=\nu_{k,l,j}$ is translation-invariant

$$\left|E_{\mu_{k,l,j}}\left[V_{k,k',l,A}\right]\right| \leq \|G\|_\infty \Big| E_{\nu_{0,l,j}}[g(\eta(0))] - E_{\nu_{j/(2l+1)}}[g(\eta(0))]\Big|.$$

By equivalence of ensembles (cf. p.355, [17]), the right hand side of the above disaplay vanishes as $l\to\infty$, uniformly for $\rho = j/(2l+1)\in[0,A]$. The lemma now is proved by letting $\gamma\to\infty$. □

8.2. **Local 2-block estimate.** We now detail the local 2-block estimate following the outline of the local 1-block estimate. Recall the notation $\Lambda_{k,l}$ from the 1-block estimate and let $\Lambda_{k,k',l}=\Lambda_{k,l}\cup\Lambda_{k',l}$ for $|k-k'|>l$. Define the generator $L_{k,k',l}$ on $\Omega_{k,k',l}=\mathbb{N}_0^{\Lambda_{k,k',l}}$:

$$\begin{aligned} L_{k,k',l}f(\eta) &= L_{k,l}f(\eta) + L_{k',l}f(\eta) \\ &\quad + g(\eta(k+l))\left[f\left(\eta^{k+l,k'-l}\right) - f(\eta)\right] + g(\eta(k'-l))\left[f\left(\eta^{k'-l,k+l}\right)-f(\eta)\right]. \end{aligned}$$

When $|k-k'|$ is large, the process governed by $L_{k,k',l}$ in effect treats the blocks $\Lambda_{k,l}$ and $\Lambda_{k',l}$ as adjacent, with a connecting bond.

Let $\Omega_{k,k',l,j} := \{\eta \in \Omega_{k,k',l} : \sum_{x\in\Lambda_{k,k',l}} \eta(x) = j\}$. As before, the localized measure $\nu^{\rho}_{k,k',l}$ defined by $\nu_\rho$ limited to sites in $\Lambda_{k,k',l}$, as well as $\nu_{k,k',l,j}$, the canonical measure of $\nu^{\rho}_{k,k;,l}$ on $\Omega_{k,k',l,j}$, are both invariant and reversible with respect to $L_{k,k',l}$.

The corresponding Dirichlet form, with measure $\kappa$ given by $\mu_{k,k',l}$ or $\mu_{k,k',l,j}$, is given by

$$\begin{aligned} E_\kappa\left[f(-L_{k,k',l}f)\right] = &\sum_{x,x+1\in\Lambda_{k,k',l}} E_\kappa\Big[g(\eta(x))\left[f\left(\eta^{x,x+1}\right)-f(\eta)\right]^2\Big] \\ &+ E_\kappa\Big[g(\eta(k+l))\Big[f\left(\eta^{k+l,k'-l}\right)-f(\eta)\Big]^2\Big]. \end{aligned} \tag{8.6}$$

For $l, j \geq 1$, let $b_{l,l,j}$ be the spectral gap of $-L_{k,k',l}$ on $\Omega_{k,k',l,j}$, cf. (8.2). As $b_{l,l,j}$ is strictly postive, we have the following Poincaré inequality (cf. (8.3)): for all $f \in L^2(\nu_{k,k',l,j})$

$$\mathrm{Var}_{\nu_{k,k',l,j}}(f) \leq C_{l,l,j} E_{\nu_{k,k',l,j}}\left[f(-L_{k,k',l}f)\right] \tag{8.7}$$

where $C_{l,l,j} := b^{-1}_{l,l,j}$ for $j \geq 1$ and $C_{l,l,0} = 0$.

We now state and show a local 2-blocks estimate. The scheme is similar to that of the local 1-block estimate.

**Lemma 8.2** (Local 2-block estimate)**.** *We have*

$$\limsup_{l\to\infty}\limsup_{\theta\to 0}\limsup_{N\to\infty}\sup_{k,j}\mathbb{E}_N\Big[\int_0^T\left|\Phi(\eta^l_s(k))-\Phi(\eta^{\theta N}_s(j))\right|ds\Big]=0 \tag{8.8}$$

*where the* sup *is taken over all $k$ and $j$ such that $k \in \Lambda_{j,\theta N}$ and $\Lambda_{j,\theta N+l} \cap \mathfrak{D}_N = \emptyset$.*

*Proof.* We separate the argument into steps.

*Step 1.* Since $\Phi(\cdot)$ is Lipschitz, to prove the lemma, it suffices to show (8.8) with $\Phi(\eta^l_s(k)) - \Phi(\eta^{\theta N}_s(j))$ replaced by $\eta^l_s(k) - \eta^{\theta N}_s(j)$. We may further replace $\eta^{\theta N}_s(j)$ by $(2\theta N+1)^{-1}\sum_{k'\in\Lambda_{j,\theta N}} \eta^l(k')$. Indeed, the replacement error is

$$\mathbb{E}_N\Big[\Big|\eta^{\theta N}_s(j) - \frac{1}{2\theta N+1}\sum_{k'\in\Lambda_{j,\theta N}}\eta^l_s(k')\Big|\Big] \leq \frac{2l+1}{2\theta N+1}\mathbb{E}_N\left[\eta^l_s(j-\theta N)+\eta^l_s(j+\theta N)\right].$$

As $\mu^N_s \leq \mathscr{R}^N_c$, by attractiveness (as explained in Step 1 of the proof of the local 1-block Lemma 8.1 either case $g(n) \sim n^\alpha$ or $g$ bounded), the expectation term $\mathbb{E}_N$ is bounded uniformly in $t$, $l$, and $N$, the right-hand side of the above display vanishes as $N \uparrow \infty$ first.

Therefore, the lemma will follow if we show

$$\limsup_{l\to\infty}\limsup_{\theta\to 0}\limsup_{N\to\infty}\sup_{k,k'}\mathbb{E}_N\Big[\int_0^T\Big|\eta^l_s(k)-\eta^l_s(k')\Big|ds\Big]=0 \tag{8.9}$$

where the sup is taken over all $k, k'$ such that $\Lambda_{k,k',l} \cap \mathfrak{D}_N = \emptyset$ and $2l+1 \leq k'-k \leq \theta N$.

*Step 2.* By a similar coupling argument as in the Step 1 of the proof of local 1-block Lemma 8.1, we may apply a cutoff of large densities. Therefore, to prove the lemma, it suffices to show

$$\limsup_{l\to\infty}\limsup_{\theta\to 0}\limsup_{N\to\infty}\sup_{k,k'}\mathbb{E}_N\Big[\int_0^T\Big|\eta^l_s(k)-\eta^l_s(k')\Big|\mathbb{1}_{\{\eta^l_s(k,k')\leq A\}}ds\Big]=0$$

where $\eta^l_s(k,k') = \eta^l_s(k) + \eta^l_s(k')$ and the sup is over $k, k'$ as in (8.9).

Let $U_{k,k',l,A}(\eta) := \left|\eta^l(k) - \eta^l(k')\right| \mathbb{1}_{\{\eta^l(k,k')\le A\}}$. Following the proof of Lemma 8.1, for fixed $l, \theta, N, k, k'$, in order to estimate $\mathbb{E}_N\left[\int_0^T U_{k,k',l,A}(\eta_s)ds\right]$, it suffices to bound

$$(\gamma N)^{-1}\lambda_{N,l} = \sup_f \left\{ E_{\mathscr{R}^N_{c_0}}\left[U_{k,k',l,A} f\right] - \gamma^{-1} N E_{\mathscr{R}^N_{c_0}}\left[\sqrt{f}(-L_N\sqrt{f})\right]\right\}. \tag{8.10}$$

where the supremum is over all $f$ which are densities with respect to $\mathscr{R}^N_{c_0}$.

*Step 3.* Recall the generator $L_{k,k',l}$ and its Dirichlet form defined in the beginning of this subsection. We now argue the following Dirichlet form inequality

$$E_{\mathscr{R}^N_{c_0}}\left[\sqrt{f}(-L_{k,k',l}\sqrt{f})\right] \le \theta N E_{\mathscr{R}^N_{c_0}}\left[\sqrt{f}(-L_N\sqrt{f})\right]. \tag{8.11}$$

The Dirichlet form with respect to the full generator $L_N$ under $\mathscr{R}^N_{c_0}$ is given by

$$E_{\mathscr{R}^N_{c_0}}\left[f(-L_N f)\right] = \sum_{k\in\mathbb{T}_N} E_{\mathscr{R}^N_{c_0}}\Big[g_{k,N}(\eta(k))\big(f(\eta^{k,k+1}) - f(\eta)\big)^2\Big]. \tag{8.12}$$

First, writing out the Dirichlet form in (8.6), in terms of the product measure $\mathscr{R}^N_{c_0}$, we have

$$\begin{aligned} E_{\mathscr{R}^N_{c_0}}\left[f(-L_{k,k',l}f)\right] = & \sum_{x,x+1\in\Lambda_{k,k',l}} E_{\mathscr{R}^N_{c_0}}\Big[g(\eta(x))\big(f(\eta^{x,x+1}) - f(\eta)\big)^2\Big] \\ & + E_{\mathscr{R}^N_{c_0}}\Big[g(\eta(k+l))\big(f(\eta^{k+l,k'-l}) - f(\eta)\big)^2\Big]. \end{aligned}$$

Next, by adding and subtracting at most $\theta N$ terms, we have

$$\left[f\left(\eta^{k+l,k'-l}\right) - f(\eta)\right]^2 \le (k'-k-2l)\sum_{q=0}^{k'-k-2l-1}\left[f\left(\eta^{k+l,k+l+q+1}\right) - f\left(\eta^{k+l,k+l+q}\right)\right]^2.$$

By the change of variables $\xi = \eta^{k+l,k+l+q}$, which takes away a particle at $k+l$ and adds one at $k+l+q$, we have $\mathscr{R}^N_{c_0}(\eta) = \dfrac{g(\eta(k+l+q)+1)}{g(\eta(k+l))}\mathscr{R}^N_{c_0}(\xi)$. Then

$$\begin{aligned} & E_{\mathscr{R}^N_{c_0}}\Big[g(\eta(k+l))\left[f\left(\eta^{k+l,k+l+q+1}\right) - f\left(\eta^{k+l,k+l+q}\right)\right]^2\Big] \\ =& \sum_\xi \mathscr{R}^N_{c_0}(\eta) g(\eta(k+l))\left[f\left(\xi^{k+l+q,k+l+q+1}\right) - f\left(\xi\right)\right]^2 \\ =& E_{\mathscr{R}^N_{c_0}}\Big[g(\eta(k+l+q))\Big[f\left(\eta^{k+l+q,k+l+q+1}\right) - f\left(\eta\right)\Big]^2\Big]. \end{aligned}$$

From these observations, (8.11) follows.

*Step 4.* Let $\mu_{k,k',l}$ be the restriction of $\mu = \mathscr{R}^N_{c_0}$ to $\Lambda_{k,k',l}$. Clearly, $\mu_{k,k',l} = \nu_{k,k',l}$. Inputting (8.11) into (8.10), and considering the conditional expectation of $f$ with respect to $\Omega_{k,k',l}$ as in the 1-block estimate proof, we have

$$(\gamma N)^{-1}\lambda_{N,l} \le \sup_{f_{k.k',l}}\left\{E_{\mu_{k,k',l}}\left[U_{k,k',l,A} f_{k,k',l}\right] - \frac{1}{\theta\gamma}E_{\mu_{k,k',l}}\Big[\sqrt{f_{k,k',l}}(-L_{k,k',l}\sqrt{f_{k,k',l}})\Big]\right\},$$

where the supremum is over densities $f_{k,k',l}$ with respect to $\mu_{k,k',l}$.

Again, as in the proof of the 1-block estimate, decomposing $f_{k,k',l}d\mu_{k,k'.l}$ along configurations with common total number $j$, we need only to bound

$$\sup_{0\le j\le A(2l+1)}\sup_f\left\{E_{\nu_{k,k',l,j}}\left[U_{k,k',l,A}f\right] - \frac{1}{\theta\gamma}E_{\nu_{k,k',l,j}}\left[\sqrt{f}(-L_{k,k',l}\sqrt{f})\right]\right\},$$

where the supremum is over densities $f$ with respect to $\nu_{k,k',l,j}$.

*Step 5.* Consider the centered object

$$\widehat{U}_{k,k',l,A} = U_{k,k',l,A} - E_{\nu_{k,k',l,j}}\left[U_{k,k',l,A}\right].$$

Recall $C_{l,l,j}$, the inverse spectral gap of $L_{k,k',l}$ from (8.7) and note that $\|\widehat{U}_{k,k',l,A}\|_\infty \leq A$. Using the Rayleigh expansion (cf. p.375, [17]), we have

$$\begin{aligned}
&E_{\nu_{k,k',l,j}}\left[\widehat{U}_{k,k',l,A} f\right] - (\theta\gamma)^{-1} E_{\nu_{k,k',l,j}}\left[\sqrt{f}(-L_{k,k',l}\sqrt{f})\right] \\
\leq &\frac{\theta\gamma}{1-2AC_{l,l,j}\,\theta\gamma} E_{\nu_{k,k;,l,j}}\left[\widehat{U}_{k,k',l,A}(-L_{k,k',l})^{-1}\widehat{U}_{k,k',l,A}\right] \\
\leq &\frac{\theta\gamma C_{l,l,j}}{1-2AC_{l,l,j}\,\theta\gamma} E_{\nu_{k,k',l,j}}\left[\widehat{U}^2_{k,k',l,A}\right] \to 0 \text{ as } \theta \to 0.
\end{aligned}$$

*Step 6.* Recall the definition of $U_{k,k',l,A}$ in Step 2. To finish, we still need to estimate the centering term $E_{\nu_{k,k',l,j}}\left[U_{k,k',l,A}\right]$. By adding and subtracting $j/(4l+2)$, we need only bound $E_{\nu_{k,k',l,j}}\left[\left|\eta^l(k) - j/(4l+2)\right|\right]$. By exchangeability and an equivalence of ensemble estimate (cf. p. 355 [17]), the canonical variance

$$\begin{aligned}
E_{\nu_{k,k',l,j}}\left[\left|\eta^l(k) - j/(4l+2)\right|^2\right] &= O(l^{-1}) E_{\nu_{k,k',l,j}}\left[(\eta(k) - j/(4l+2))^2\right] \\
&+ O(1) E_{\nu_{k,k',l,j}}\left[(\eta(k) - j/(4l+2))(\eta(k+1) - j/(4l+2))\right]
\end{aligned}$$

and is further bounded by $C(A)\mathrm{Var}_{\nu^{j/(4l+2)}_{k,k',l}}\left(\eta^l(k)\right)$ for some constant $C(A)$ depending only on $A$. This variance is of order $O(l^{-1})$, since the single site variance $\mathrm{Var}_{\nu^{j/(4l+2)}_{k,k',l}}(\eta(k))$ is uniformly bounded for $j/(4l+2) \leq A$. Hence, $\sup_{0\leq j\leq A(4l+2)} E_{\nu_{k,k',l,j}}\left[V_{k,k',l,A}\right]$ is of order $O(l^{-1/2})$, vanishing as $l \uparrow \infty$. This finishes the proof. $\square$

**Remark 8.3.** *We comment that Lemmas 8.1 and 8.2 will hold with the same argument if $\eta^l(x)$, and similarly $\eta^{\theta N}(x)$, is replaced with a nearby average $\eta^{l,+}(x) = \frac{1}{2l+1}\sum_{y=x+1}^{x+2l+1}(x)$. This will be useful in treating 'replacement' near the boundary of a defect site in Section 9.*

8.3. **Bulk Replacement Lemma.** Let $G(t,x)$ be a bounded function on $[0,T]\times\mathbb{T}$ with compact support on $[0,T]\times(\mathbb{T}\setminus\mathfrak{D})$. Lemma 8.1 implies that

$$\limsup_{l\to\infty}\limsup_{N\to\infty}\mathbb{E}_N\Big[\frac{1}{N}\sum_{k\in\mathbb{T}_N}\Big|\int_0^T G(s,k/N)\left(g(\eta(k)) - \Phi(\eta^l(k))\right)ds\Big|\Big] = 0$$

and by Lemma 8.2,

$$\limsup_{l\to\infty}\limsup_{\theta\to 0}\limsup_{N\to\infty}\mathbb{E}_N\Big[\frac{1}{N}\sum_{k\in\mathbb{T}_N}\int_0^T G(s,k/N)\Big|\Phi\left(\eta^l_s(k)\right) - \Phi\left(\eta^{\theta N}_s(k)\right)\Big|ds\Big] = 0.$$

By Markov's inequality and triangle inequality, we obtain

**Lemma 8.4** (Bulk Replacement Lemma)**.** *For each bounded function $G(t,x)$ on $[0,T]\times\mathbb{T}$ with compact support on $[0,T]\times(\mathbb{T}\setminus\mathfrak{D})$, and $\delta>0$, we have*

$$\limsup_{\theta\to 0}\limsup_{N\to\infty}\mathbb{P}_N\Big[\Big|\frac{1}{N}\sum_{k\in\mathbb{T}_N}\int_0^T G\big(t,\frac{k}{N}\big)\Big(g\left(\eta_t(k)\right) - \Phi\left(\eta^{\theta N}_t(k)\right)\Big)dt\Big| \geq \delta\Big] = 0.$$

**Remark 8.5.** *We comment, in the $g(n) \sim n^\alpha$ setting (where 'FEM' as stated in* [17] *holds), the "attractiveness" assumption used in the local* 1 *and* 2*-block estimates to introduce cutoffs of large densities in a local region, may be dropped in the statement of Lemma 8.4 about a global average.*

## 9. Replacement at the boundary

In this section, we show a local replacement near the defect sites in two steps. In Lemma 9.1, we show that the jump rate $g_{k,N(\eta_s(k))}$ at any site $k \in \mathbb{T}_N$ is close to $g_{k+1,N(\eta_s(k+1))}$, that of its neighbor site $k+1$. When $k$ is a defect site in $\mathfrak{D}_N$, this neighbor site will be a non-defect or regular site for $N$ large. Then, we may apply local 1 and 2-blocks estimates from last section to obtain our local replacement Lemma 9.2 near the defect sites.

**Lemma 9.1.** *Let $G$ be any continuous function on $[0,T]$. Then, for any $j \in J_s \cup J_c$ and $k = k_{j,N}$, we have*

$$\limsup_{N\to\infty} \mathbb{E}_N \Big[ \Big| \int_0^T G(s)\big(g_{k,N}(\eta_s(k)) - g_{k+1,N}(\eta_s(k+1))\big) ds \Big| \Big] = 0.$$

*Proof.* The following argument will hold in both $k^\alpha$ and $g$ bounded settings. Fix $j \in J_s \cup J_c$. Let

$$U_s(\eta) = 2N^{-1}(G(s))^2\big(g_{k,N}(\eta(k)) + g_{k+1,N}(\eta(k+1))\big).$$

By Lemma 5.2, $\lim_{N\to\infty} \sup_{k\in\mathbb{T}_N} \mathbb{E}_N\big[\int_0^T U_s(\eta_s)ds\big] = 0$. Let

$$V_s(\eta) = G(s)\big(g_{k,N}(\eta(k)) - g_{k+1,N}(\eta(k+1))\big).$$

Then, to prove the lemma, it suffices to show that

$$\limsup_{\kappa\to\infty} \limsup_{N\to\infty} \mathbb{E}_N \Big[ \Big| \int_0^T V_s(\eta_s) ds \Big| - \kappa \int_0^T U_s(\eta_s) ds \Big] = 0. \tag{9.1}$$

As $H(\mu^N | \mathscr{R}_c^N) \le CN$ for $c = c_0$, by the entropy inequality, the expectaton in the previous display is bounded from above by

$$\frac{C}{\kappa} + \frac{1}{\kappa N} \ln \mathbb{E}_{\mathscr{R}_c^N} \Big[ \exp\Big\{ \kappa N \Big| \int_0^T V_s(\eta_s)ds \Big| - \kappa^2 N \int_0^T U_s(\eta_s)ds \Big\} \Big].$$

The absolute value in the right hand side of last inequality can be dropped by using $e^{|x|} \le e^x + e^{-x}$. By Feynman-Kac formula (cf. p.336, [17]), we have

$$\frac{1}{\kappa N} \ln \mathbb{E}_{\mathscr{R}_c^N} \Big[ \exp\Big\{ \kappa N \int_0^T (V_s - \kappa U_s) ds \Big\} \Big] \le \frac{1}{\kappa N} \int_0^T \lambda_N(s) ds$$

where $\lambda_N(s)$ is the largest eigenvalue of $N^2 L_N + \kappa N(V_s(\eta) - \kappa U_s(\eta))$. Fix $s \in [0,T]$ and note the variational formula for $\lambda_N$:

$$(\kappa N)^{-1}\lambda_N = \sup_f \Big\{ E_{\mathscr{R}_c^N}\left[(V_s - \kappa U_s)f\right] - \kappa^{-1} N E_{\mathscr{R}_c^N}\big[\sqrt{f}(-L_N\sqrt{f})\big] \Big\}$$

where the supremum is over all $f$ which are densities with respect to $\mathscr{R}_c^N$ (cf. [17], p. 377). Thus, to prove (9.1), it remains to show, for any density $f$,

$$E_{\mathscr{R}_c^N}[V_s f] \le E_{\mathscr{R}_c^N}[\kappa U_s f] + \kappa^{-1} N E_{\mathscr{R}_c^N}\big[\sqrt{f}(-L_N\sqrt{f})\big]. \tag{9.2}$$

By the product structure of $\mathscr{R}_c^N$, we have

$$g_{k,N}(\eta(k))\mathscr{R}_c^N(\eta) = g_{k+1,N}(\eta(k+1)+1)\mathscr{R}_c^N(\eta^{k,k+1}).$$

Thus, we compute that

$$
\begin{aligned}
E_{\mathscr{R}_c^N}[V_s f] =& E_{\mathscr{R}_c^N}\Big[G(s)\Big(g_{k,N}(\eta(k)) - g_{k+1,N}(\eta(k+1))\Big)f(\eta)\Big] \\
=& E_{\mathscr{R}_c^N}\Big[G(s)g_{k,N}(\eta(k))\Big(f(\eta) - f(\eta^{k,k+1})\Big)\Big] \\
=& E_{\mathscr{R}_c^N}\Big[G(s)g_{k,N}(\eta(k))\Big(\sqrt{f(\eta)} - \sqrt{f(\eta^{k,k+1})}\Big)\Big(\sqrt{f(\eta)} + \sqrt{f(\eta^{k,k+1})}\Big)\Big].
\end{aligned}
$$

By Cauchy-Schwarz, for any $A > 0$, the above display is estimated from above by

$$
\begin{aligned}
&AE_{\mathscr{R}_c^N}\Big[g_{k,N}(\eta(k))\Big(\sqrt{f(\eta)} - \sqrt{f(\eta^{k,k+1})}\Big)^2\Big] \\
&\qquad\qquad + A^{-1}E_{\mathscr{R}_c^N}\Big[G(s)^2 g_{k,N}(\eta(k))\Big(\sqrt{f(\eta)} + \sqrt{f(\eta^{k,k+1})}\Big)^2.
\end{aligned}
$$

Notice that the first expectation in the above display is bounded by $E_{\mathscr{R}_c^N}\big[\sqrt{f}(-L_N\sqrt{f})\big]$, cf. (8.12).. Take $A = \kappa^{-1}N$. The second summand is estimated from above by

$$
\begin{aligned}
&2\kappa N^{-1}E_{\mathscr{R}_c^N}\Big[(G(s))^2 g_{k,N}(\eta(k))\big(f(\eta) + f(\eta^{k,k+1})\big)\Big] \\
=& 2\kappa N^{-1}E_{\mathscr{R}_c^N}\Big[(G(s))^2\big(g_{k,N}(\eta(k)) + g_{k+1,N}(\eta(k+1))\big)f(\eta)\Big] \\
=& E_{\mathscr{R}_c^N}\big[\kappa U_s f\big].
\end{aligned}
$$

Retracing the terms, we obtain (9.2), finishing the proof. □

We now finish this section with a local replacement lemma at defect sites:

**Lemma 9.2** (Local replacement at defect sites)**.** *Let $G$ be any continuous function on $[0,T]$. Then, for each defect site $k_{j,N} \in \mathfrak{D}_N$, we have*

$$
\lim_{\theta\to 0}\limsup_{N\to\infty}\mathbb{E}_N\left[\left|\int_0^T G(s)\left(g_{k_{j,N},N}(\eta_s(k_{j,N})) - \Phi(\eta_s^{\theta N,+}(k_{j,N}))\right)ds\right|\right] = 0
$$

*where $\eta^{l,+}(k) := (2l+1)^{-1}\sum_{k+1\le x\le k+2l+1}\eta(x)$.*

*Proof.* Lemma 9.1 shows we may replace $g_{k_{j,N},N}(\eta_s(k_{j,N}))$ by $g_{k_{j,N}+1,N}(\eta_s(k_{j,N}+1))$. To finish, notice $k_{j,N}+1 \in \mathbb{T}_N \setminus \mathfrak{D}_N$ for $N$ large. Then, we may further replace $g_{k_{j,N}+1,N}(\eta_s(k_{j,N}+1))$ by $\Phi(\eta_s^{\theta N,+}(k_{j,N}))$ using Lemma 8.1 and Lemma 8.2; see Remark 8.3. The proof is now complete. □

## 10. Energy estimate

By Lemma 7.1, we know that $Q$ is supported on paths $\pi_t(dx)$ which can be decomposed into an absolute continuous part $\rho(t,x)dx$ and atoms $\sum_{j\in J_c}\mathfrak{m}_j(t)\delta_{x_j}(dx)$. In this section, we prove an energy estimate for $\rho(t,x)$.

**Proposition 10.1.** *$Q$ is supported on paths $\pi_t(dx)$ such that $\Phi(\rho(t,x))$ is weakly differentiable with respect to $x$ on $[0,T]\times\mathbb{T}$ and $\partial_x\Phi(\rho(t,x))$, the weakly derivative, satisfies*

$$
\int_0^T\int_{\mathbb{T}}\frac{|\partial_x\Phi(\rho(t,x)|^2}{\Phi(\rho(t,x)}dx\,dt < \infty.
$$

*Proof.* The proof presented here is based on the one of Theorem 7.1, p. 102, [17]. However, because of the presence of the slow site and the difference in the underlining topology, many details are different in subtle ways.

*Step 1.* Let $\{H_j\}_{j\in\mathbb{N}}$ be a dense sequence in $C_c^{0,1}([0,T]\times(\mathbb{T}\setminus\mathfrak{D}))$ under the norm $\|H\|_\infty+\|\partial_x H\|_\infty$. Recall the constant $K_0$ as in the relative entropy bound $H(\mu^N|\mathscr{R}_c^N)\le K_0N$. In the first two steps, we show that, for all $\epsilon$ small,

$$\limsup_{N\to\infty}\mathbb{E}_N\Big[\max_{1\le j\le m}\Big\{\int_0^T W_N(\epsilon,H_j(s,\cdot),\eta_s)ds\Big\}\Big]\le K_0 \tag{10.1}$$

where

$$\begin{aligned}W_N(\epsilon,H(\cdot),\eta):=&\sum_{x\in\mathbb{T}_N}\frac{H(x/N)}{\epsilon N}\left(g_{x,N}(\eta(x))-g_{x+\epsilon N,N}(\eta(x+\epsilon N))\right)\\&-\frac{2}{N}\sum_{x\in\mathbb{T}_N}\frac{H^2(x/N)}{\epsilon N}\sum_{0\le k\le\epsilon N}g_{x+k,N}(\eta(x+k))\end{aligned}$$

By the entropy inequality, the expectation in (10.1) is bounded from above by

$$\frac{1}{N}H(\mu^N|\mathscr{R}_c^N)+\frac{1}{N}\ln\mathbb{E}_{\mathscr{R}_c^N}\Big[\exp\Big\{\max_{1\le j\le m}\Big\{N\int_0^T W_N(\epsilon,H_j(s,\cdot),\eta_s)ds\Big\}\Big\}\Big].$$

Using the relative entropy bound $H(\mu^N|\mathscr{R}_c^N)\le K_0N$ and the inequality $e^{\max a_j}\le\sum e^{a_j}$, the previous display is estimated from above by

$$K_0+\max_{1\le j\le m}\limsup_{N\to\infty}\frac{1}{N}\ln\mathbb{E}_{\mathscr{R}_c^N}\Big[\exp\Big\{N\int_0^T W_N(\epsilon,H_j(s,\cdot),\eta_s)ds\Big\}\Big].$$

By Feynman-Kac formula, for any fixed index $j$, the limsup term in previous expression is less than or equal to

$$\limsup_{N\to\infty}\int_0^T\sup_f\Big\{E_{\mathscr{R}_c^N}\left[W_N(\epsilon,H_j(s,\cdot),\eta)f(\eta)\right]-NE_{\mathscr{R}_c^N}\Big[\sqrt{f}(-L_N\sqrt{f})\Big]\Big\}ds$$

where the supremum is over all $f$ which are densities with respect to $\mathscr{R}_c^N$.

*Step 2.* To show (10.1), it now remains to show, for all $H$ in $C_c^{0,1}([0,T]\times(\mathbb{T}\setminus\mathfrak{D}))$, that

$$E_{\mathscr{R}_c^N}\left[W_N(\epsilon,H(s,\cdot),\eta)f(\eta)\right]-NE_{\mathscr{R}_c^N}\Big[\sqrt{f}(-L_N\sqrt{f})\Big]\le 0. \tag{10.2}$$

We first compute that $E_{\mathscr{R}_c^N}\left[W_N(\epsilon,H(s,\cdot),\eta)f(\eta)\right]$ equals

$$\begin{aligned}&E_{\mathscr{R}_c^N}\Big[\sum_{x\in\mathbb{T}_N}\frac{H(x/N)}{\epsilon N}(g_{x,N}(\eta(x))-g_{x+\epsilon N,N}(\eta(x+\epsilon N))f(\eta)\Big]\\&-\frac{2}{N}E_{\mathscr{R}_c^N}\Big[\sum_{x\in\mathbb{T}_N}\frac{H^2(x/N)}{\epsilon N}\sum_{0\le k\le\epsilon N}g_{x+k,N}(\eta(x+k))\Big].\end{aligned} \tag{10.3}$$

Let $\delta_x$ be the configuration with the only particle at $x$ and $\eta+\delta_x$ be the configuration obtaining from adding one particle at $x$ to $\eta$ By the definition of $\mathscr{R}_c^N$, we have, for each $x$,

$$E_{\mathscr{R}_c^N}[g_{x,N}(\eta(x))f(\eta)]=\varphi E_{\mathscr{R}_c^N}[f(\eta+\delta_x)] \tag{10.4}$$

where $\varphi=\Phi(c)$. Then, the first expectation in (10.3) is written as

$$\sum_{x\in\mathbb{T}_N}\frac{\varphi H(x/N)}{\epsilon N}E_{\mathscr{R}_c^N}\Big[f(\eta+\delta_x)-f(\eta+\delta_{x+\epsilon N})\Big]. \tag{10.5}$$

which is rewritten as

$$E_{\mathscr{R}_c^N}\Big[\sum_{x\in\mathbb{T}_N}\sum_{0\le k\le\epsilon N-1}\frac{\varphi H(x/N)}{\epsilon N}\big(\sqrt{f(\eta+\delta_{x+k})}+\sqrt{f(\eta+\delta_{x+k+1})}\big)\\ \times\big(\sqrt{f(\eta+\delta_{x+k})}-\sqrt{f(\eta+\delta_{x+k+1})}\big)\Big]. \tag{10.6}$$

Using $2ab\le a^2+b^2$, for any $A>0$, (10.6) is bounded from above by

$$E_{\mathscr{R}_c^N}\Big[\sum_{x\in\mathbb{T}_N}\sum_{0\le k\le\epsilon N-1}\frac{\varphi H^2(x/N)}{2\epsilon NA}\big(\sqrt{f(\eta+\delta_{x+k})}+\sqrt{f(\eta+\delta_{x+k+1})}\big)^2\Big]\\ +E_{\mathscr{R}_c^N}\Big[\sum_{x\in\mathbb{T}_N}\sum_{0\le k\le\epsilon N-1}\frac{\varphi A}{2\epsilon N}\big(\sqrt{f(\eta+\delta_{x+k})}-\sqrt{f(\eta+\delta_{x+k+1})}\big)^2\Big]. \tag{10.7}$$

The second expectation in (10.7) is recognized as

$$E_{\mathscr{R}_c^N}\Big[\sum_{x\in\mathbb{T}_N}\frac{A}{2}\big(\sqrt{f(\eta+\delta_x)}-\sqrt{f(\eta+\delta_{x+1})}\big)^2\Big]=\frac{A}{2}E_{\mathscr{R}_c^N}\Big[\sqrt{f}(-L_N\sqrt{f})\Big].$$

For the first expectation in (10.7), using first $(\sqrt{a}+\sqrt{b})^2\le 2(a+b)$ and then the change of variable formula (10.4), it is bounded from above by

$$\sum_{x\in\mathbb{T}_N}\frac{2H^2(x/N)}{\epsilon NA}\sum_{0\le k\le\epsilon N}E_{\mathscr{R}_c^N}\Big[g_{x+k,N}(x+k)f(\eta)\Big]. \tag{10.8}$$

Notice that the summation of $k$ is ranging from $0\le k\le\epsilon N$ instead of $0\le k\le\epsilon N-1$. Now, we set $A=N$. Putting together (10.3) and (10.8), we obtain (10.2).

*Step 3.* Recall that $H_j$'s have compact support in $[0,T]\times(\mathbb{T}\setminus\mathfrak{D})$. Appling the Bulk Replacement Lemma (Lemma 8.4) to (10.1) and taking $N\to\infty$, we obtain

$$\limsup_{\delta\to 0}E_Q\Big[\max_{1\le j\le m}\Big\{\int_0^T\int_{\mathbb{T}}H_j(s,x)\epsilon^{-1}\big(\Phi\langle\iota_\delta(\cdot-x),\pi_s\rangle-\Phi\langle\iota_\delta(\cdot-x-\epsilon),\pi_s\rangle\big)dxds\\ -2\int_0^T\int_{\mathbb{T}}H_j^2(s,x)\epsilon^{-1}\int_x^{x+\epsilon}\Phi\langle\iota_\delta(\cdot-u),\pi_s\rangle dudxds\Big\}\Big]\le K_0,$$

Sending $\delta\to 0$, applying a discrete integration by parts, and then taking $\epsilon\to 0$, we have

$$E_Q\Big[\max_{1\le j\le m}\Big\{\int_0^T\int_{\mathbb{T}}\partial_x H(s,x)\Phi(\rho(s,x))dxds\\ -2\int_0^T\int_{\mathbb{T}}H^2(s,x)\Phi(\rho(s,x))dxds\Big\}\Big]\le K_0,$$

By monotone convergence, the $\max_{1\le j\le m}$ above can be replaced by $\max_{1\le j<\infty}$. Furthermore, as $H_j$ is dense in $C_c^{0,1}([0,T]\times(\mathbb{T}\setminus\mathfrak{D}))$ with the norm $\|H\|_\infty+\|\partial_x H\|_\infty$, we conclude

$$E_Q\Big[\sup_H\Big\{\int_0^T\int_{\mathbb{T}}\partial_x H(s,x)\Phi(\rho(s,x))dxds-2\int_0^T\int_{\mathbb{T}}H^2(s,x)\Phi(\rho(s,x))dxds\Big\}\Big]\le K_0, \tag{10.9}$$

where the sup is over $H\in C_c^{0,1}([0,T]\times(\mathbb{T}\setminus\mathfrak{D}))$.

*Step 4.* As a result of (10.9), for $Q$-a.e. path $\pi_t(dx)$, there exists $B = B(\pi_t)$ such that, for all $C_c^{0,1}([0,T] \times (\mathbb{T} \setminus \mathfrak{D}))$,

$$\int_0^T \int_{\mathbb{T}} \partial_x H(s,x)\Phi(\rho(s,x))dxds - 2\int_0^T \int_{\mathbb{T}} H^2(s,x)\Phi(\rho(s,x))dxds \le B.$$

Define on $C_c^{0,1}([0,T] \times D_n)$ a linear functional $l(H) := \int_0^T \int_{\mathbb{T}} \partial_x H(s,x)\Phi(\rho(s,x))dxds$. Also define $\|H\|_{2,\rho} = \left(\int_0^T \int_{\mathbb{T}} H^2(s,x)\Phi(\rho(s,x))dxds\right)^{1/2}$. Then we have, for all $a \in \mathbb{R}$

$$a\, l(H) - 2a^2 \left(\|H\|_{2,\rho}\right)^2 \le B.$$

Maximizing the left hand side over $a \in \mathbb{R}$, we obtain $l(\cdot)$ is a bounded linear functional. By Riesz representation theorem, there exists $F$ such that $l(H) = \langle H, F\rangle$ and $\|F\|_{2,\rho} < \infty$. Define $\partial_x \Phi(\rho(t,x)) = -F(t,x)\Phi(\rho(t,x))$. Then, we have shown that $\Phi(\rho(t,x))$ is weakly differentialble with respect to $x$ on $[0,T] \times (\mathbb{T} \setminus \mathfrak{D})$. As both $(\partial_x \Phi(\rho))^2/\rho$ and $\rho$ are in $L^1([0,T] \times \mathbb{T})$, by Cauchy-Schwarz, we have $\partial_x \Phi(\rho)$ is in $L^1([0,T] \times \mathbb{T})$ as well. Then, we conclude that $\lim_{x\to x_j+} \Phi(\rho(t,x))$ and $\lim_{x\to x_j-} \Phi(\rho(t,x))$ exists and are finite for all $x_j \in \mathfrak{D}$. Furthermore, by Lemma 9.2, the left and right limits match at all $x_j$. This extends the weak differentiability of $\Phi(\rho)$ from $\mathbb{T} \setminus \mathfrak{D}$ to $\mathbb{T}$ and the proposition is now proved. $\square$

## 11. Uniqueness

In this section, we present the uniqueness of the weak solutions to equations (4.1) and (4.2). The proof is based on an energy argument (cf. [16]).

**Theorem 11.1.** *There exists at most one weak solution to* (4.1).

*Proof.* Let $\pi_t^{(1)}$ and $\pi_t^{(2)}$ be two weak solutions of (4.1) such that $\pi_t^{(i)} = \rho_i(t,x)dx + \sum_{j\in J_c} \mathfrak{m}_j^{(i)}(t)\delta_{x_j}(dx)$ for $i = 1, 2$. As $\mathfrak{m}_j^{(i)}(t) = \left[\lambda_j \Phi(\rho_i(t,x_j))\right]^{1/\alpha}$, to prove the theorem, it suffices to show $\rho_1 = \rho_2$.

*Step 1.* Define $\overline{\Phi}(t,x) := \Phi(\rho_1(t,x)) - \Phi(\rho_2(t,x))$. We first show that $\overline{\Phi}$ can be approximated "well" by a squence $\{\overline{\Phi}_\varepsilon\}_{\varepsilon>0}$ in the sence that (1) $\overline{\Phi}_\varepsilon$ is smooth and compactly supported on $[0,T] \times (\mathbb{T} \setminus \mathfrak{D}_s)$; (2) $\overline{\Phi}_\varepsilon \to \overline{\Phi}$ and $\partial_x \overline{\Phi}_\varepsilon \to \partial_x \overline{\Phi}$ in $L^2([0,T] \times \mathbb{T})$; and (3) $\overline{\Phi}_\varepsilon(t,x_j) \to \overline{\Phi}(t,x_j)$ in $L^2[0,T]$ for all $j \in J_c$ as $\varepsilon \to 0$.

For $\delta > 0$, let $\mathfrak{D}_s^\delta = \cup_{j\in J_s}(x_j - \delta, x_j + \delta)$. We define $F_\delta$ be such that $F_\delta(t,x) = 9$ on $[0,T] \times \mathfrak{D}_s^\delta$ and $F_\delta(t,x) = \overline{\Phi}(t,x)$ on $[0,T] \times (\mathbb{T} \setminus \mathfrak{D}_s^{2\delta})$. For $(t,x) \in \mathfrak{D}_s^{2\delta}$, let

$$F_\delta(t,x) = \begin{cases} \overline{\Phi}(t, 2x - x_j - 2\delta) & [0,T] \times [x_j + \delta, x_j + 2\delta), \\ \overline{\Phi}(t, 2x - x_j + 2\delta) & [0,T] \times (x_j - 2\delta, x_j - \delta]. \end{cases}$$

As $\partial_x \overline{\Phi} \in L^2([0,T] \times \mathbb{T})$ and $\overline{\Phi}(t,x_j) = 0$ for $j \in J_s$, we have that $F_\delta$ and $\partial_x F_\delta$ approximates $\overline{\Phi}$ and $\partial_x \overline{\Phi}$ in $L^2([0,T] \times \mathbb{T})$ respectively. Notice that $F_\delta(t,x_j) = \overline{\Phi}(t,x_j)$ for all $\delta$ small and $j \in J_c$. To find a desired sequence $\{\overline{\Phi}_\varepsilon\}$, by a diagonal argument, it suffices to show that, for each small $\delta$, there exist $\{F_{\delta,\varepsilon}\}$ that approximates $F_\delta$ "well".

To this end, let $\tau_\varepsilon(x)$ be the standard mollifier supported on $[-\varepsilon, \varepsilon]$. With $F_\delta(t,x)$ extended to be 0 for $t \notin [0,T]$, we define

$$F_{\delta,\varepsilon}(t,x) := \int_{\mathbb{R}} \int_{\mathbb{T}} F_\delta(t-s, x-u)\tau_\varepsilon(s)\tau_\varepsilon(u)duds$$

Then $F_{\delta,\varepsilon} \in C_c^\infty(\mathbb{R} \times (\mathbb{T} \setminus \mathfrak{D}_s))$ for $\varepsilon < \delta$. When restricted on $[0,T] \times \mathbb{T}$, it is standard that $F_{\delta,\varepsilon}$ and $\partial_x F_{\delta,\varepsilon}$ approximates $F_\delta$ and $\partial_x F_\delta$ respectively in $L^2([0,T] \times \mathbb{T})$ as $\varepsilon \to 0$. For the approximation of $F_\delta(t,x_j)$ by $F_{\delta,\varepsilon}(t,x_j)$ in $L^2[0,T]$, notice that

$$\begin{aligned}&\int_0^T (F_{\delta,\varepsilon}(t,x_j) - F_\delta(t,x_j))^2\, dt \\ &\qquad = \int_0^T \Big[ \int_{\mathbb{R}} \int_{\mathbb{T}} (F_\delta(t-s, x_j - u) - F_\delta(t,x_j))\, \tau_\varepsilon(s)\tau_\varepsilon(u) du ds\Big]^2 dt.\end{aligned}$$

By adding and subtracting $F_\delta(t-s,x_j)$, the above is bounded above by $I_1 + I_2$ where

$$\begin{aligned}I_1 &:= 2\int_0^T \Big[ \int_{\mathbb{R}} \int_{\mathbb{T}} (F_\delta(t-s, x_j - u) - F_\delta(t-s,x_j))\, \tau_\varepsilon(s)\tau_\varepsilon(u) du ds\Big]^2 dt, \\ I_2 &:= 2\int_0^T \Big[ \int_{\mathbb{R}} (F_\delta(t-s, x_j) - F_\delta(t,x_j))\, \tau_\varepsilon(s) ds\Big]^2 dt\end{aligned}$$

As $\int_{\mathbb{R}} F_\delta(t-s,x_j)\tau_\varepsilon(s)ds$ approximates $F_\delta(t,x_j)$ in $L^2[0,T]$, the term $I_2$ vanishes as $\varepsilon \to 0$. For the term $I_1$, using $F_\delta(t-s,x_j-u) - F_\delta(t-s,x_j) = \int_{x_j}^{x_j-u} \partial_x F_\delta(t-s,x)dx$, we have

$$I_1 \le 2\int_0^T \int_{\mathbb{R}} \int_{\mathbb{T}} \Big( \int_{x_j}^{x_j-u} \partial_x F_\delta(t-s,x) dx\Big)^2 \tau_\varepsilon(s)\tau_\varepsilon(u) du ds dt$$

which is further bounded by $2\varepsilon^2 \int_0^T \int_{\mathbb{T}} |\partial_x F_\delta(t,x)|^2 dx dt \to 0$ as $\varepsilon \to 0$.

*Step 2.* We now proceed to the uniqueness of weak solutions. Let $\overline{\rho} := \rho_1 - \rho_2$ and $\overline{\mathfrak{m}}_j(t) := \mathfrak{m}_j^{(1)}(t) - \mathfrak{m}_j^{(2)}(t)$ for each $j \in J_c$. As $\pi_t^{(1)}$ and $\pi_t^{(2)}$ both satisfy (4.3), we have, for all $G(t,x) \in C_c^\infty([0,T) \times (\mathbb{T} \setminus \mathfrak{D}_s))$,

$$\int_0^T \int_{\mathbb{T}} \partial_t G(t,x)\overline{\rho}(t,x) dx dt + \sum_{j\in J_c} \int_0^T \partial_t G(t,x_j) \overline{\mathfrak{m}}_j(t) dt = \int_0^T \int_{\mathbb{T}} \partial_x G(t,x) \partial_x \overline{\Phi}(t,x) dx dt. \tag{11.1}$$

Let $\overline{\Phi}$ be approximated "well" by some $\{\overline{\Phi}_\varepsilon\}$ as in Step 1. Taking $G(t,x) = -\int_t^T \overline{\Phi}_\varepsilon(s,x)ds$ and then leting $\varepsilon \to 0$, we obtain

$$\begin{aligned}&\int_0^T \int_{\mathbb{T}} \overline{\Phi}(t,x)\overline{\rho}(t,x) dx dt + \sum_{j\in J_c} \int_0^T \overline{\Phi}(t,x_j)\overline{\mathfrak{m}}_j(t) dt \\ &\qquad = -\int_{\mathbb{T}} \int_0^T \Big[ \int_t^T \partial_x \overline{\Phi}(s,x) ds\Big] \partial_x \overline{\Phi}(t,x) dt dx.\end{aligned} \tag{11.2}$$

The right part of the above is computed as $-\dfrac{1}{2}\int_{\mathbb{T}} \big[ \int_0^T \partial_x \overline{\Phi}(t,x)dt\big]^2 dx \le 0$. However, for the left part, it holds $\overline{\Phi}(t,x)\overline{\rho}(t,x) \ge 0$ and $\overline{\Phi}(t,x_j)\overline{\mathfrak{m}}_j(t) \ge 0$ for all $t$, $j$, and $x$. Then we have that $\int_0^T \int_{\mathbb{T}} \overline{\Phi}(t,x)\overline{\rho}(t,x) dx dt = 0$ which implies $\overline{\Phi}(t,x)\overline{\rho}(t,x) = 0$ a.e. and, therefore, $\overline{\rho}(t,x) = 0$ a.e.. The theorem is proved. □

**Theorem 11.2.** *There exists at most one weak solution to* (4.2).

*Proof.* For $i = 1,2$, let $\pi_t^{(i)} = \rho_i(t,x)dx + \sum_{j\in J_c} \mathfrak{m}_j^{(i)}(t)\delta_{x_j}(dx)$ be two weak solutions to (4.2). Notice that $\mathfrak{m}_j^{(i)}(t) = \mathfrak{m}_j^{(i)}(t)\mathbb{1}_{\rho(t,x_j)=c_j}$ and $\rho(t,x_j) \le c_j$ implies $\overline{\Phi}(t,x_j)\overline{\mathfrak{m}}_j(t) \ge 0$ where $\overline{\Phi} := \Phi(\rho_1) - \Phi(\rho_2)$ and $\overline{\mathfrak{m}}_j(t) := \mathfrak{m}_j^{(1)}(t) - \mathfrak{m}_j^{(2)}(t)$. Following proof of Theorem 11.1, we have $\rho_1 = \rho_2$. To conclude the theorem, it remains to show $\mathfrak{m}_j^{(1)} = \mathfrak{m}_j^{(2)}$ for each $j$. To this end, for each $j \in J_c$, take $F \in C^\infty(\mathbb{T})$ such that $F(x_j) = 1$ and $\operatorname{supp} F \cap J_c = x_j$. Letting $G(t,x) = \int_t^T h(s)F(x)ds$ in (11.1) for any $h(t) \in C_c^\infty(0,T)$, we obtain $\int_0^T h(t)\overline{\mathfrak{m}}_j(t)dt = 0$, and therefore, $\overline{\mathfrak{m}}_j(t) = 0$ a.e. for all $j$. The proof is now complete. $\square$

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