Exam 1

Math 125, Section 8, Fall 2005

September 14, 2005

For full credit, show all your work/sketch any graphs that you produce on your calculators.

Problem 1 20 points

Air pressure decays exponentially as a function of height above sea-level at a *continuous rate* of 12% per kilometer.

(a) Write a formula for the air pressure as a function of h, the height above sea-level in kilometers.

If P_0 is the pressure at sea level, the pressure $P(h)$ as a function of height h is given by

$$P(h) = P_0 e^{-0.12h}.$$

(b) What is the air pressure at the top of Mt. McKinley, height of 6198 meters as a percentage of the sea-level air pressure. (Note:1 Kilometer = 1000 meters)

Clearly, 6198 meters = 6.198 kilometers. Using the formula from the previous part, we get

$$P = P_0 e^{-0.12 \times 6.198} = 0.475 P_0$$

Consequently, the pressure at the top of Mt. McKinley is 47.5pressure at sea level.

(c) What is the *halving-height* for the air pressure, that is what is the height at which the air pressure is half it's sea level value?

The halving height is determined by requiring that the pressure be $P_0/2$. Using this in the above equation, we get

$$\frac{P_0}{2} = P_0 e^{-0.12h}.$$

Rearranging and taking natural logarithms of both sides yields

$$h = \frac{\ln(0.5)}{0.12} = 5.775\text{Km}$$

Problem 2 20 points

Gallileo discovered that the period of a swinging pendulum is **proportional** to the square-root of the length of the pendulum. When the pendulum is two feet long, the period is 1.57 seconds.

(a) Find the equation for the period of a pendulum, P, in terms of its length, l.

We are given that the period is proportional to the square-root of the length.

Consequently,

$$P = k\sqrt{l}$$

where k is a constant.

If $l = 2$ ft, $P = 1.57$ seconds. using this information, we get

$$k = \frac{1.57s}{\sqrt{2\text{ft}}} = 1.11 second/\sqrt{ft}.$$

If P is measured in seconds and l in feet, we have

$$P = 1.11\sqrt{l}.$$

(b) Foucault demonstrated rotation of the earth by hanging a pendulum from the roof of a building in Paris named Pantheon. The pendulum takes 16.5 seconds for an oscillation. Find the height of Pantheon, assuming the pendulum at rest is one foot above the ground of the building.

Using the prrevious part, along with $P = 16.5$, we see

$$16.5 = 1.11\sqrt{l}$$

where l is measured in feet.

Solving for l gives $l = 220.9$ ft. This is the length of the pendulum.

Adding to this, the 1 ft for the separation between the bottom of the pendulum and the floor, we see that the height of the pantheon is 221.9 ft.

Problem 3 40 points.

(a) Solve the equation $P_0 a^t = Q_0 b^t$ for t assuming P_0, Q_0, a, b are positive constants.

Taking natural logs on both sides, we get

$$\ln(P_0) + t\ln(a) = \ln(Q_0) + t\ln(b).$$

rearranging and solving for t yields,

$$t = \frac{\ln P_0 - \ln Q_0}{\ln(b) - \ln(a)}.$$

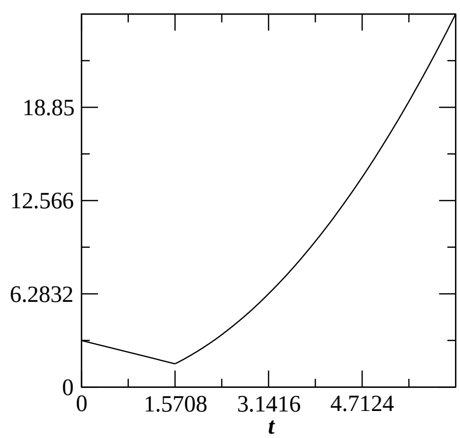
(b) A particle moves on a circle of radius 5 and the angle θ between the position of the particle and the x-axis is given by

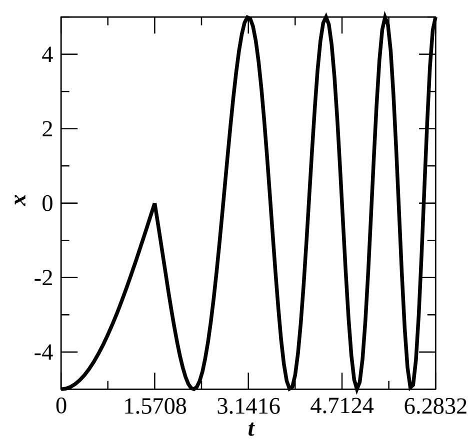
$$\theta = \begin{cases} \pi - t & \text{for } 0 \le t < \pi/2 \\ \frac{2}{\pi}t^2 & \text{for } \pi/2 < t \le 2\pi \end{cases}$$

Graph the x-coordinate as a function of time.

For a circle with radius r, the x-coordinate is given by $x = r\cos(\theta)$.

Consequently, $x = 5\cos(\theta)$ where θ is given by the above function.





Plotting θ as a function of t, we see that θ decreases from π to $\pi/2$ and then increases until 8π. So, the point on the circle starts at the negative x-axis, moves clockwise for a quarter revolution, and then moves counter clockwise for 3 and 3/4 revolutions, ending up at $\theta = 8\pi$. I've sketched two graphs below, one for θ as a function of t, and the other for x as a function of t.

x	1.9	1.99	1.999	2.001	2.01	2.1
$\frac{\sin(\pi x)}{x-2}$	3.090	3.141	3.14159	3.14159	3.1411	3.090

(c) Let $f(x) = \dfrac{\sin(\pi x)}{x-2}$. Using a table of values estimate $L = \lim_{x\to 2} f(x)$, and find a value δ such that $|x-2| < \delta$ guarantees that $|f(x) - L| < 0.001$.

Looking at the table, we estimate that

$$\lim_{x\to 2} f(x) = \pi = 3.141592...$$

If $|f(x) - \pi| < 0.001$, it follows that $\pi - 0.001 < f(x) < \pi + 0.001$ which is $3.1405 < f(x) < 3.1425$.

From the table, we see that if $1.99 \leq x \leq 2.01$, $f(x)$ is in the range of $3.1405 < f(x) < 3.1425$. Consequently, $|x-2| \leq 0.01 \Rightarrow 1.99 \leq x \leq 2.01$ is an appropriate range of x-values that is, $\delta = 0.01$.

(d) Find the limit

$$\lim_{x\to\infty} \frac{ax^3 - 4x^2 + 2}{1 + 8x^k}$$

or state that the limit does not exist. (Hint: The answer depends on a and k, and you have to do separate cases)

If $a = 0$, the leading power in the numerator is 2, otherwise it is 3. So, we do the cases $a = 0$ and $a \neq 0$ separately.

If $a = 0$, we have the following cases:

$$k > 2 : \lim_{x\to\infty} \frac{ax^3 - 4x^2 + 2}{1 + 8x^k} = \lim_{x\to\infty} \frac{x^2\left(-4 + \frac{2}{x^2}\right)}{x^k\left(8 + \frac{1}{x^k}\right)} = \lim_{x\to\infty} \frac{x^{2-k}\left(-4 + \frac{2}{x^2}\right)}{\left(8 + \frac{1}{x^k}\right)} = 0$$

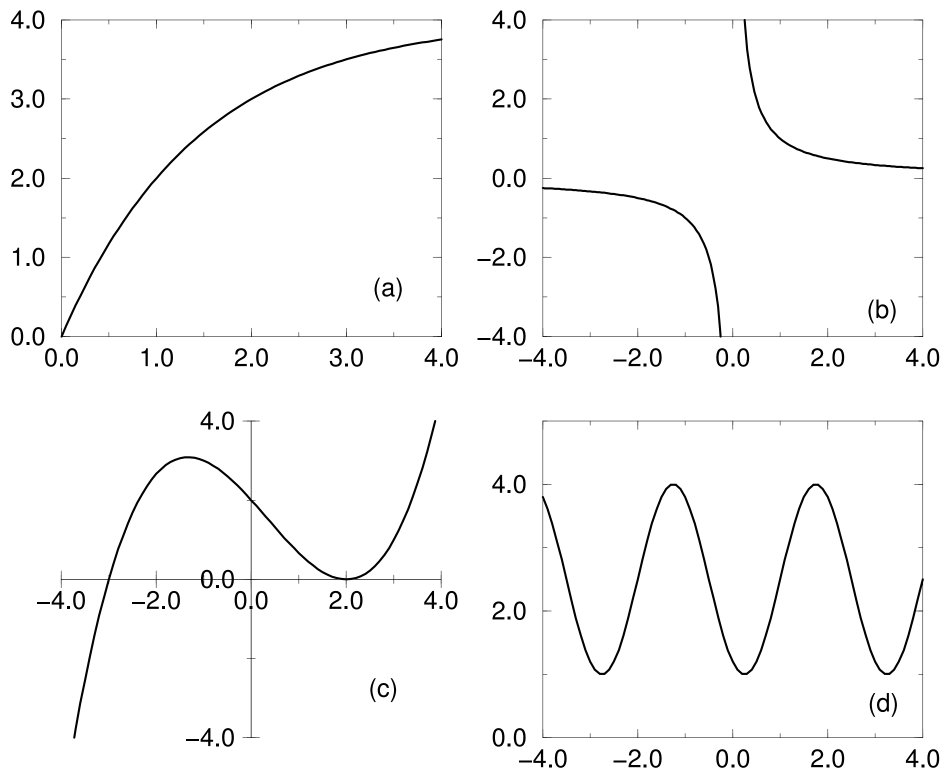
$$k = 2 : \lim_{x\to\infty} \frac{ax^3 - 4x^2 + 2}{1 + 8x^k} = \lim_{x\to\infty} \frac{x^2}{x^2}\frac{\left(-4 + \frac{2}{x^2}\right)}{\left(8 + \frac{1}{x^2}\right)} = -\frac{4}{8} = \frac{1}{2}$$

$k < 2$:The numerator grows faster $\Rightarrow$ limit does not exist

Similarly, for $a \neq 0$, we have 3 cases:

$$k > 3 : \lim_{x \to \infty} \frac{ax^3 - 4x^2 + 2}{1 + 8x^k} = 0$$
$$k = 3 : \lim_{x \to \infty} \frac{ax^3 - 4x^2 + 2}{1 + 8x^k} = \frac{a}{8}$$
$$k < 3 : \lim_{x \to \infty} \frac{ax^3 - 4x^2 + 2}{1 + 8x^k} \text{ does not exist.}$$

Problem 4 Find possible formulas for the graphs in the following figure. 20 points.



(a) We see that the first function saturates to the value $y = 4$. Consequently, $4 - y$ is an exponential function, that is $4 - y = P_0 a^x$.

At $x = 0$, $y = 0$, so that $4 - 0 = P_0 a^0 = P_0$ giving $P_0 = 4$.

At $x = 1$, $y = 2$, so that $4 - 2 = 4a^1$ giving $a = 1/2$.

Combining these results we get $4 - y = 4(1/2)^x$, so that

$$y = 4\left[1 - \frac{1}{2^x}\right]$$

(b) The graph has a horizonatal asymptote at $y = 0$, and a vertical asymptote $x = 0$, and it takes both positive and negative values. A possible functional form is

$$y = \frac{1}{x}$$

(c) The graph has a double zero at $x = 2$, corresponding to a factor $(x - 2)^2$.

The graph has a simple zero at $x = -3$, corresponding to a factor $(x+3)$.

Putting these factors together, we have

$$y = k(x-2)^2(x+3)$$

At $x = 0$, we have $y = 2$. Consequently

$$2 = k(0-2)^2(0+3) = 12k,$$

yielding $k = 1/6$. Therefore a formula which describes this graphs is

$$y = \frac{(x-2)^2(x+3)}{6}.$$

(d) The function is periodic.

Maximum value = 4, minimum = 1.

Consequently, average = (4+1)/2 = 5/2, amplitude = (4-1)/2 = 3/2.

The function crosses the average value, going upwards at $x = -2, 1$ and 4. Consequently the period of the function is 3.

Putting all this information together, we see that

$$y = \frac{5}{2} + \frac{3}{2}\sin\left[\frac{2\pi}{3}(x-1)\right]$$