SOLUTIONS TO SELECTED PROBLEMS

13.20 Let S and T be subsets of $\mathbb{R}$. Prove the following:

(a) $\mathrm{cl}(\mathrm{cl}(S)) = \mathrm{cl}(S)$.
Let $G = \mathrm{cl}(S)$. Then G is closed. It follows $\mathrm{cl}(G) = G$.

(b) $\mathrm{cl}(S \cup T) = \mathrm{cl}(S) \cup \mathrm{cl}(T)$.

Proof. We start by proving an alternate characterization of $\mathrm{cl}(S)$.

Claim $x \in \mathrm{cl}(S)$ iff for all $\epsilon > 0$, $N(x;\epsilon) \cap S \neq \emptyset$.

Given $\epsilon > 0$. If $x \in \mathrm{cl}(S)$, then $x \in S \cup S'$.

If $x \in S$ then $x \in N(x;\epsilon) \cap S$. Also, if $x \in S'$, then $N^*(x;\epsilon) \cap S \neq \emptyset$ implying that $N(x;\epsilon) \cap S \neq \emptyset$. In either case, we see that $x \in \mathrm{cl}(S) \implies \forall\epsilon > 0\ N(x;\epsilon) \cap S \neq \emptyset$.

Conversely, assume that $\forall\epsilon > 0\ N(x;\epsilon) \cap S \neq \emptyset$. If $x \in S$, then $x \in \mathrm{cl}(S)$ so we only need to consider the case $x \notin S$. If $x \notin S$ and $\forall\epsilon > 0\ N(x;\epsilon) \cap S \neq \emptyset$, it follows that $\forall\epsilon > 0\ N^*(x;\epsilon) \cap S \neq \emptyset$. This shows that $x \in S' \subseteq \mathrm{cl}(S)$. This proves that, if for all $\epsilon > 0$, $N(x;\epsilon) \cap S \neq \emptyset$, then $x \in \mathrm{cl}(S)$.

We now use this characterization to prove that $\mathrm{cl}(S \cup T) = \mathrm{cl}(S) \cup \mathrm{cl}(T)$.

If $x \in \mathrm{cl}(S) \cup \mathrm{cl}(T)$, then $x \in \mathrm{cl}(S)$ or $x \in \mathrm{cl}(T)$.

If $x \in \mathrm{cl}(S)$, it follows that $\forall\epsilon > 0\ N(x;\epsilon) \cap S \neq \emptyset$, which implies $\forall\epsilon > 0\ N(x;\epsilon) \cap (S \cup T) \neq \emptyset$, showing that $x \in \mathrm{cl}(S \cup T)$. This proves $\mathrm{cl}(S) \subseteq \mathrm{cl}(S \cup T)$.

A similar argument also shows $\mathrm{cl}(T) \subseteq \mathrm{cl}(S \cup T)$, implying that $\mathrm{cl}(S) \cup \mathrm{cl}(T) \subseteq \mathrm{cl}(S \cup T)$.

We will prove the converse through the contrapositive. Assume that $x \notin \mathrm{cl}(S) \cup \mathrm{cl}(T)$. Then $x \notin \mathrm{cl}(S)$ and $x \notin \mathrm{cl}(T)$.

Since $x \notin \mathrm{cl}(S)$, it follows that there exists an $\epsilon_1 > 0$ such that $N(x;\epsilon_1) \cap S = \emptyset$. Similarly, $x \notin \mathrm{cl}(T)$, so there exists an $\epsilon_2 > 0$ such that $N(x;\epsilon_2) \cap T = \emptyset$.

Set $\delta = \min(\epsilon_1, \epsilon_2)$. Then $\delta > 0$, and $N(x;\delta) \cap S = \emptyset$, $N(x;\delta) \cap T = \emptyset$ showing that $N(x;\delta) \cap (S \cup T) = \emptyset$. This proves $x \notin \mathrm{cl}(S) \cup \mathrm{cl}(T) \implies x \notin \mathrm{cl}(S \cup T)$. Taking the contrapositive implies $\mathrm{cl}(S \cup T) \subseteq \mathrm{cl}(S) \cup \mathrm{cl}(T)$. $\square$

(c) $\mathrm{cl}(S \cap T) \subseteq \mathrm{cl}(S) \cap \mathrm{cl}(T)$.

Proof. If $x \in \mathrm{cl}(S \cap T)$, it follows that $\forall\epsilon > 0\ N(x;\epsilon) \cap S \cap T \neq \emptyset$, which implies $\forall\epsilon > 0\ N(x;\epsilon) \cap S \neq \emptyset$ and $\forall\epsilon > 0\ N(x;\epsilon) \cap T \neq \emptyset$, showing that $x \in \mathrm{cl}(S) \cap \mathrm{cl}(T)$. This proves $\mathrm{cl}(S \cap T) \subseteq \mathrm{cl}(S) \cap \mathrm{cl}(T)$. $\square$

(d) Find an example to show that equality need not hold in (c).
Let $S = (0, 1)$ and $T = (1, 2)$. Then $\mathrm{cl}(S) \cap \mathrm{cl}(T) = \{1\} \neq cl(S \cap T) = \mathrm{cl}(\emptyset) = \emptyset$.

13.22 For any set $S \subseteq \mathbb{R}$, let $\bar{S}$ denote the intersection of all the closed sets containing S.
Let $\mathcal{F}$ denote thate family of all the closed sets containing S, *i.e.*,

$$\mathcal{F} = \{F \subseteq \mathbb{R} \mid S \subseteq F, F \text{ is closed.}\}$$

Then

$$\bar{S} = \bigcap_{F \in \mathcal{F}} F.$$

(a) Prove that $\bar{S}$ is a closed set.
$\bar{S}$ is an intersection of closed sets, and is hence closed.

(b) Prove that $\bar{S}$ is the smallest closed set containing S.

Proof. By definition of the family $\mathcal{F}$, we see that $x \in S \implies x \in F \ \forall F \in \mathcal{F}$. Consequently, $x \in S \implies x \in \cap_{F \in \mathcal{F}} F = \bar{S}$, so that $S \subseteq \bar{S}$.

Also, $y \in \bar{S} \implies y \in F \ \forall F \in \mathcal{F}$. If C is a closed set containing S, it follows that $C \in \mathcal{F}$, so that $y \in \bar{S} \implies y \in C$, so that $\bar{S} \subseteq C$. □

(c) Prove that $\bar{S} = \text{cl}(S)$.

Proof. Since cl$S$ is a closed set containing S. it follows from (b) that $\bar{S} \subseteq \text{cl}(S)$.

We will prove the opposite inclusion by proving it's contrapositive. Assume that $x \notin \bar{S}$. By de Morgan's law, it follows that

$$x \in \bigcup_{F \in \mathcal{F}} \mathbb{R} \backslash F.$$

By the definition of union over an indexed family, we see that, there exists a set $G \in \mathcal{F}$ such that $x \in U = \mathbb{R} \backslash G$. Since G is closed, it follows that $U = \mathbb{R} \backslash G$ is open. Since $x \in U$ which is open, there exists an $\epsilon > 0$ such that $N(x; \epsilon) \subseteq U$, which implies $\exists \epsilon > 0 \ni N(x; \epsilon) \cap G = \emptyset$. Since $S \subseteq G$, it also follows that $\exists \epsilon > 0 \ni N(x; \epsilon) \cap S = \emptyset$. From our characterization of the closure of S in Problem 13.20, it follows that $x \notin \bar{S} \implies x \notin \text{cl}(S)$, so that, $\text{cl}(S) \subseteq \bar{S}$. □

(d)If S is bounded, show that $\bar{S}$ is also bounded.

Proof. If S is bounded, there exists $M < \infty$ such that $x \in S \implies |x| \leq M$. This implies $x \in [-M, M]$ so that $S \subseteq [-M, M]$.

$[-M, M]$ is therefore a closed set containing S, and by part (b), we can conclude that $\bar{S} \subseteq [-M, M]$. This proves that $\bar{S}$ is also bounded. □

14.8 If S is a compact subset of $\mathbb{R}$ and T is a closed subset of S, then T is compact .

(a) Prove this using the definition of compactness.

Proof. We are given $\mathcal{G}$, a collection of open sets that covers T and from $\mathcal{G}$, we want to extract a finite subcover for T.

Let $U = \mathbb{R}\backslash T$. Clearly, $\mathcal{G} \cup \{U\}$ is a cover of all of $\mathbb{R}$ by open sets since every element of $\mathcal{G}$ is open, $U = \mathbb{R}\backslash T$ is open, and every element of $\mathbb{R}$ is either in T so that it is covered by $\mathcal{G}$ or in $\mathbb{R}\backslash T = U$.

Since S is a compact subset of $\mathbb{R}$, it follows that we can extract a finite subcover from $\mathcal{G} \cup \{U\}$ that covers S. There are two cases to consider:

Case 1 : The finite subcover does not contain U. Then $S \subseteq G_1 \cup G_2 \cup \ldots \cup G_n$ where each $G_i \in \mathcal{G}$. Since $T \subseteq S$, it follows that $T \subseteq G_1 \cup G_2 \cup \ldots \cup G_n$ where each $G_i \in \mathcal{G}$, thereby yielding a finite subcover for T.

Case 2: The finite subcover does contain U. Then $S \subseteq G'_1 \cup G'_2 \cup \ldots \cup G'_m \cup U$ where each $G'_j \in \mathcal{G}$. Since $T \subseteq S$, it follows that $T \subseteq G'_1 \cup G'_2 \cup \ldots \cup G'_m \cup U$. We are not done yet, since this is not a finite subcover of T from the cover $\mathcal{G}$. In particular the cover contains the extraneous element U.

Now we note that $U = \mathbb{R}\backslash T$ so that $x \in T \implies x \notin U$. Therefore

$$x \in T \implies x \in G'_1 \cup G'_2 \cup \ldots \cup G'_m \cup U \implies x \in G'_1 \cup G'_2 \cup \ldots \cup G'_m$$

since $x \notin U$. Consequently, $T \subseteq G'_1 \cup G'_2 \cup \ldots \cup G'_m$ where each $G'_j \in \mathcal{G}$, thereby yielding a finite subcover for T. □

(b) Prove this using the Heine-Borel theorem.

Proof. S is compact, so it is closed and bounded. Since $T \subseteq S$, it follows that T is also bounded. By the hypothesis of the problem, T is closed. This implies T is compact. □

16.12 (a) Suppose that $\lim s_n = 0$. If (t_n) is a bounded sequence, prove that $\lim s_n t_n = 0$.

Proof. Since t_n is a bounded sequence, $\exists M < \infty$ such that $|t_n| < M$ for all $n \in \mathbb{N}$. Given $\epsilon > 0$, since $\lim s_n = 0$, it follows that $\exists N$ such that for all $n \geq N$, $|s_n| < \frac{\epsilon}{M+1}$.

Consequently, for $n \geq M$, it follows that

$$|s_n t_n - 0| < \frac{\epsilon M}{M+1} < \epsilon$$

proving that $\lim s_n t_n = 0$. □

(b) Show by example that the boundedness of t_n is necessary in (a)

Let $s_n = 1/n$, $t_n = 2n$. Then $s_n t_n = 2$ for all n and it does not converge to zero, although $\lim s_n = 0$.

16.15 (a) Prove that x is an accumulation point of S iff there exists a sequence (s_n) of points in $S\backslash\{x\}$ such that (s_n) converges to x.

Proof. Assume that (s_n) is a sequence in $S\backslash\{x\}$ that converges to x. Let $\epsilon > 0$ be arbitrary. Since $\lim s_n = x$, it follows that there exists N such that for all $n \geq N$, $|s_n - x| < \epsilon$. In particular, there is a natural number M such that $N < M \leq N+1$ and $|s_M - x| < \epsilon$. Also, since the sequence (s_n) is in $S\backslash\{x\}$, it follows that $s_M \neq x$. Therefore, s_M is in the deleted neighborhood $N^*(x;\epsilon)$ intersected with the set S.

We have thus shown, $\forall \epsilon > 0, N^*(x;\epsilon) \cap S \neq \emptyset$, proving that x is an accumulation point.

Conversely, assume that x is an accumulation point for S. This implies, for each $n \in \mathbb{N}$, $\mathbb{N}^*(x; 1/n) \cap S \neq \emptyset$. Consequently, we can pick a point $s_n \in \mathbb{N}^*(x; 1/n) \cap S$.

By construction $s_n \in S\backslash\{x\}$ and $|s_n - x| < 1/n$. This implies that there exists a sequence (s_n) of points in $S\backslash\{x\}$ such that (s_n) converges to x. □

(b) Prove that a set S is closed iff whenever (s_n) is a convergent sequence of points in S, it follows that $\lim s_n$ is in S.

Proof. Assume that S is not closed, so that S' is not contained in S. Then, there exists $x \in S'$ such that $x \notin S$. Since $x \notin S$, it follows that $S\backslash\{x\} = S$. By the previous part, it follows that there is a convergent sequence (s_n) in S such that $\lim s_n = x$, that is, there is a convergent sequence in S whose limit is not in S. Taking the contrapositive, we see that, if all convergent sequences in S have limits in S, then S contains all its accumulation points, so that S is closed.

Conversely, assume that S is closed and (s_n) is a convergent sequence in S. If $\lim s_n$ is in S, there is nothing more to show. If the $\lim s_n = x \notin S$, then (s_n) is a convergent sequence in $S\backslash\{x\}$. By the previous part, it follows that x is an accumulation point of S. Since S is closed, $x \in S$ contradicting the assumption that $x \notin S$. This proves that if S is closed, whenever (s_n) is a convergent sequence of points in S, it follows that $\lim s_n$ is in S. □

17.18 Suppose that (s_n) is a convergent sequence with $a \leq s_n \leq b$ for all $n \in \mathbb{N}$. Prove that $a \leq \lim s_n \leq b$.

Proof. This is the "slick" proof. Let $S = [a, b]$. Then S is closed and (s_n) is a convergent sequence in a closed set. It follows from Problem 16.15 (b) that $\lim s_n \in S = [a, b]$.

The "transparent and detailed" proof is as follows. Assume that s_n converges to l and $l \notin S = [a.b]$. Then $l \in \mathbb{R}\backslash[a, b]$ which is an open set. Consequently, there exists an $\epsilon > 0$ such that $N(l;\epsilon) \cap [a, b] = \emptyset$. Since $s_n \in [a, b]\ \forall n \in \mathbb{N}$, it follows that $s_n \notin N(l;\epsilon)$ for any $n \in N$. This of course contradicts the claim that s_n converges to $l \notin S$. This proves $a \leq \lim s_n \leq b$. □