

Take home final
Math 323, Section 3, Fall 2008

You are not allowed to discuss these problems with anyone else. Also, you are not allowed to use any sources besides your text book and your class notes. You are allowed to quote theorems from the book, but not the results of homework problems, which is you use, you must prove again.

All problems carry equal weight. Please write up your work clearly, starting each problem on a new page and when you turn in the exam, please make sure that the problems are in order.

1. Given sets A and B Let $M(A, B)$ denote the set of all functions with domain A and co-domain B . We can define a relation \mathbf{R} on $M(A, B)$ by $f : A \rightarrow B, g : A \rightarrow B$ and

$$f \mathbf{R} g \iff \forall D \subseteq B \text{ we have } f[g^{-1}(D)] \subseteq D.$$

Prove or give a counterexample for each of the following statements:

- (a) \mathbf{R} is a reflexive relation.
- (b) \mathbf{R} is a transitive relation.
- (c) \mathbf{R} is a symmetric relation.

(Hint: Show that $f \mathbf{R} g$ if and only if $\forall D \subseteq B$ we have $g^{-1}(D) \subseteq f^{-1}(D)$.)

2. A set $I \subseteq \mathbb{R}$ is called an *interval*, if it has the following property:

$$a \in I, b \in I, a \leq c \leq b \implies c \in I.$$

- (a) Show that I can be written as a union of all the closed intervals contained in I , *i.e.*,

$$I = \bigcup_{F \in \mathcal{F}} F \quad \mathcal{F} = \{[a, b] \mid [a, b] \subseteq I\}.$$

- (b) Show that, we can find a countable (either denumerable or finite) collection of closed intervals $[a_n, b_n] \subseteq I$ such that $I = \cup_n [a_n, b_n]$. (Hint: Consider cases where I is bounded/unbounded and also contains/does not contain its sup and inf.)

- (c) Show that, I can also be written as an intersection of open intervals containing I , *i.e.*,

$$I = \bigcap_{O \in \mathcal{G}} O \quad \mathcal{G} = \{(a, b) \mid (a, b) \supseteq I\}.$$

3. Given $a \in \mathbb{N}$ define a sequence recursively by $x_0 = 1$ and

$$x_{n+1} = \frac{x_n}{2} + \frac{a}{2x_n}.$$

- (a) Show that $x_n \in \mathbb{Q}$ for all n .
(b) Show that $1 \leq x_n \leq a$ for all n .
(c) Show that there is a $\theta < 1$ such that $|x_{n+1} - x_n| \leq \theta|x_n - x_{n-1}|$ for all n .
(d) Show that x_n converges to \sqrt{a} .

4. $A \subseteq \mathbb{R}$, and A' denotes the set of all the accumulation points of A .

- (a) If $y \in A'$ and $U \subseteq \mathbb{R}$ is an open set containing y , show that there are infinitely many distinct points in $A \cap U$.
(b) Show that

$$A' = \bigcap_{x \in A} \text{cl}(A \setminus \{x\}).$$

- (c) Using this, or otherwise, show that A' is a closed set.

5. Prove or disprove the following:

- (a) A is finite and U is a open subset of \mathbb{R} . If $A \subseteq U$, there exists an $\epsilon > 0$ such that for all $x \in A$, $N(x, \epsilon) \subseteq U$.
(b) P is countable and U is a open subset of \mathbb{R} . If $P \subseteq U$, there exists an $\epsilon > 0$ such that for all $x \in P$, $N(x, \epsilon) \subseteq U$.
(c) F is closed and U is a open subset of \mathbb{R} . If $F \subseteq U$, there exists an $\epsilon > 0$ such that for all $x \in F$, $N(x, \epsilon) \subseteq U$.
(d) K is compact and U is a open subset of \mathbb{R} . If $K \subseteq U$, there exists an $\epsilon > 0$ such that for all $x \in K$, $N(x, \epsilon) \subseteq U$.

Extra credit

1. The modified Dirichlet function is defined by

$$f(x) = \begin{cases} \frac{1}{q} & x = \frac{p}{q} \text{ in lowest terms} \\ 0 & \text{otherwise} \end{cases}$$

- (a) Show that f is discontinuous at every rational number.
- (b) Show that f is continuous at every irrational number.
- (c) Does the limit $\lim_{x \rightarrow 1/2} f(x)$ exist? Prove your claim. If the limit exists, evaluate the limit.