Math 527A – Fall 06
Homework 1 : Due Sep. 12

1.1 A fundamental axiom for the real numbers $\mathbb{R}$ is that every bounded subset of $\mathbb{R}$ has a least upper bound or *supremum.*

In what follows, $\{x_n\}$ is a bounded sequence, that is, there is a bound $C < \infty$ such that $|x_n| \le C$ for all n. Let $l = \sup_n x_n$.

(a) Show that, for all $\epsilon > 0$, there is an index m such that $x_m > l - \epsilon$.

(b) Show that $l \le C$.

(c) Find, with proof, $\sup_n x_n^2$.

(d) If x_n is a monotone nondecreasing sequence, show that $\lim_{n\to\infty} x_n$ exists.

(e) Prove this proposition from class: $a_n \ge 0$ is a sequence of non-negative reals. The infinite series $\sum_{n=1}^{\infty} a_n$ converges if and only if there is constant $C < \infty$ such that for all N we have $\sum_{n=1}^{N} a_n \le C$.

1.2 (X, d_X) and (Y, d_Y) are metric spaces.

(a) If $U \subseteq X$, show that (U, d_U) is a metric space where $d_U(u_1, u_2) = d_X(u_1, u_2)$ for all $u_1, u_2 \in U$.

(b) Show that (X, ρ) is a metric space, where

$$\rho(x_1, x_2) = \frac{d_X(x_1, x_2)}{1 + d_X(x_1, x_2)}$$

(c) Show that the cartesian product $X \times Y$ is a metric space with the metric $d(z_1, z_2) = d_X(x_1, x_2) + d_Y(y_1, y_2)$ where $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$.

1.3 $\mathbf{x} \in \mathbb{R}^n$.

(a) Show that

$$\lim_{p\to\infty} \|\mathbf{x}\|_p = \max(|x_1|, |x_2|, \ldots, |x_n|).$$

(b) Show that, for all $p, q \ge 1$,

$$\frac{\|\mathbf{x}\|_p}{n} \le \|\mathbf{x}\|_q \le n\|\mathbf{x}\|_p.$$

1.4 Complete the proof of the proposition from class by showing that if $1 \le p < q \le \infty$, then $l^p(\mathbb{R}, \mathbb{N}) \subset l^q(\mathbb{R}, \mathbb{N})$, and there is a sequence $\{a_n\}$ which is an element in $l^q(\mathbb{R}, \mathbb{N})$ but not in $l^p(\mathbb{R}, \mathbb{N})$.

1.5 $0 < \alpha < 1$ and $0 < \beta < 1$. Find all $1 <\le p <\le \infty$ such that the "two-dimensional" sequence

$$x_{i,j} = \frac{1}{i^\alpha + j^\beta}$$

is an element of $l^p(\mathbb{R}, \mathbb{N}^2)$.

1.6 Let $0 < p < 1$.

(a) Show that $d_p(\mathbf{x}, \mathbf{y}) = |x - y|^p$ defines a metric on $\mathbb{R}$.

(b) Show that,

$$d_p(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i|^p$$

defines a metric on $\mathbb{R}^n$.

(c) Can this definition be generalized to sequence spaces? For each p, what is the appropriate set of sequences on which the metric is defined?