Math 527A – Fall08
Homework 10 : Due Dec. 8

10.1 Problems 3.7.32 and 3.7.33, Pg. I-170 of Prof. Flaschka's notes.

10.2 **Nonlinear functions are generally not continuous with respect to weak convergence**.

(a) Problem 3.7.34, Pg. I-170 of Prof. Flaschka's notes.

(b) If $\mathbf{x}^{(n)}$ converges weakly to $\mathbf{x}$ in $l^2(\mathbb{R}, \mathbb{N})$, show that, for each index i, $x_i^{(n)} \to x_i$.

(c) Using part (b), or otherwise, show that, for any fixed k, the nonlinear function $f_k(\mathbf{x}) = \sum_{i=1}^k x_i^2$ is a *sequentially continuous* function with respect to the weak topology.

(d) Is $\|\mathbf{x}\| = \lim_{k\to\infty} \sqrt{f_k(\mathbf{x})}$ continuous with respect to weak convergence?

10.3 $X = L^2([0, 1])$ is a normed linear space, and its dual $X^* = X$ with the usual inner product. f_n is a sequence in X that converges weakly to an element $f \in X$.

(a) Show that, for all $\epsilon > 0$, there exists an index N such that for $n \geq N$, $\|f_n\|_2 \geq \|f\|_2 - \epsilon$.

(b) If in addition, we know that $\|f_n\|_2 \to \|f\|_2$, show that $\|f_n - f\|_2 \to 0$, *i.e.* the sequence f_n converges *strongly*!

10.4 $f_\alpha : X \to \mathbb{R}$ is a family of functions from a set X into $\mathbb{R}$ that is indexed by $\alpha \in A$. For every pair of distinct elements $x \neq y$ in X, there is an index $\gamma \in A$ such that $f_\gamma(x) \neq f_\gamma(y)$.

(a) If $\mathcal{T}$ is a topology on X such that all the functions f_α are continuous, show that $\mathcal{T}$ is Hausdorff.

(b) Problem 3.7.35, Pg. I-170 of Prof. Flaschka's notes.

10.5 **Weak convergence with the energy "leaking away to infinity"**.

For all $g \in L^2(\mathbb{R})$, it can be shown that

$$\lim_{M\to\infty} \int_{-M}^{M} |g(x)|^2 dx = \|g\|_2 < \infty.$$

Use this to solve Problem 3.7.36, Pg. I-170 of Prof. Flaschka's notes.

10.6 **Is the weak topology on l^2 first countable?**

(a) Problem 3.7.38, Pg. I-171 of Prof. Flaschka's notes.

(b) Use this to show that there is no metric on $l^2(\mathbb{R}, \mathbb{N})$ that generates the weak topology.