Math 527A – Fall 08
Homework 5 : Due Oct. 20

6.1 The discrete heat equation

(a) a_i and b_i are finite sequences defined for $i = 0, 1, 2, \ldots, N$. Show the *summation by parts* formula

$$\sum_{i=0}^{N-1} (a_{i+1} - a_i) b_i = a_N b_N - a_0 b_0 - \sum_{i=1}^{N} a_i (b_i - b_{i-1})$$

The discrete equation is given by

$$\dot{u}_i = \kappa \left[\frac{u_{i+1} - u_i}{l_i} + \frac{u_{i-1} - u_i}{l_{i-1}} \right] + q_i \qquad i = 1, 2, \ldots, n-1$$
$$u_0(t) = u_n(t) = 0$$
$$u_i(0) = v_i$$

where $q_i(t), i = 1, 2, 3, \ldots, n-1$ is the forcing, and $l_i > 0$ for $i = 0, 1, 2, \ldots, n-1$ with $\sum_{i=0}^{n-1} l_i = L$.

(b) The "discrete" l^2 norm is defined by

$$\|u\|_E = \left[\sum_{i=i}^{n-1} u_i^2 l_i \right]^{1/2}.$$

Compute $\frac{d}{dt} \|u\|_E^2$.

(c) If $\{q_i\} = 0$ (no forcing!) show that $\|u\|_E \to 0$ as $t \to \infty$.

(d) Show that the discrete heat equation from above has a unique solution.

(e) Assuming that $\|u(0)\|_E = \alpha$ and $\|q(t)\|_E \leq \beta$ for all t, obtain an upper bound for $\|u(t)\|_E$ in terms of α, β, κ, L and t.

6.2 f is a function in $C^1([0,1])$ such that $f(0) = f(1) = 0$, and it's energy norm is defined by

$$\|f\|_E = \sqrt{\int_0^1 [f'(x)]^2 dx}$$

(a) Show that, there is a constant C such that

$$\|f\|_p \leq C \|f\|_E$$

for all p. Can you estimate this constant (doesn't have to be a sharp estimate)?

(b) If f is a C^1 function that is defined on the interval $[0, l]$ and vanishes at the endpoints. Generalize the above definition of the energy norm, and show that

$$\|f\|_p \leq C l^{\alpha(p)} \|f\|_E$$

where C is the same constant as above and $\alpha(p)$ is an exponent that can be obtained from dimensional analysis/rescaling.

6.3 A string of length L is clamped at its endpoints. It is under tension T and is subject to a transverse force per unit length $\tau(x)$. The equilibrium displacement $u(x)$ then satisfies the equation(s)

$$Tu''(x) = -\tau(x), \quad u(0) = u(L) = 0.$$

(a) Assuming that there does exist a C^2 solution for $u(x)$, find upper bounds for the maximum displacement and also the total energy $\mathcal{E}[u] = T\|u\|_E^2/2$ in terms of the the tension T, the length L of the string, and the root-mean-squared applied forcing

$$\bar{\tau} = \sqrt{\frac{1}{L}\int_0^L [\tau(x)]^2 dx}$$

Also, check that your results are dimensionally consistent.

(b) Consider the sequence of applied transverse forces

$$\tau_n(x) = \frac{n}{n^2(2x-L)^2+1}$$

Again, assuming that we have a sequence of C^2 solutions $u_n(x)$, show that there are constants (independent of n) $\mathcal{E}_0$ and u_{max} such that

$$\frac{T}{2}\|u_n\|_E^2 \leq \mathcal{E}_0, \qquad \|u_n\|_\infty \leq u_{max}, \quad \forall n.$$

(c) What can you say about the root-mean-squared quantities $\bar{\tau}_n$? Does this contradict the conclusions from the previous two parts?

(d) Show that the sequence of solutions $u_n(x)$ is Cauchy in the energy norm, *i.e*, given an $\epsilon > 0$, there is an index N such that for all $m, n > N$, $\|u_m - u_n\|_E < \epsilon$.

(e) What is the "physical interpretation" of the limit $n \to \infty$, and what can you conclude from the above analysis?

6.4 Assume that the heat equation, and the associated boundary conditions

$$\frac{1}{2}u_{xx} = u_t, \quad u(x,0) = u_0(x), u(0,t) = u(l,t) = 0.$$

has a smooth solution $u(x,t)$ for all smooth initial conditions $u_0(x)$.

(a) Show that the "spatial" L^2 and energy norms, decay as a function of time, and use this to prove the uniqueness of solutions of the heat equation.

(b) Show that there is a constant C such that $\|u(.,t)\|_{L^2} \leq \|u_0\|_{L^2}e^{-Ct}$. Show that the map $S_t : u_0 \mapsto u(x,t)$ is a continuous map from $L^2([0,l])$ into itself for all $t \geq 0$.

6.5 The wave equation with dissipation is given by

$$u_{xx} = u_{tt} + bu_t$$

with the initial/boundary conditions

$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x), \quad u(0,t) = u(l,t) = 0$$

Show the uniqueness of smooth solutions of this equation by constructing an appropriate energy.