

Math 527A – Fall 08
Homework 6 : Due Nov. 3

6.1 If \mathcal{T}_1 and \mathcal{T}_2 are two topologies on X , show that $(X, \mathcal{T}_1 \cap \mathcal{T}_2)$ is also a topological space. Give an example where $\mathcal{T}_1 \cup \mathcal{T}_2$ is not a topology on X .

6.2 The structure of open sets in \mathbb{R} .

In what follows, we are considering the standard (metric) topology on \mathbb{R} .

- (a) Let S be a nonempty open subset of \mathbb{R} . For each $x \in S$, let $A_x = \{a \in \mathbb{R} : (a, x] \subseteq S\}$ and $B_x = \{b \in \mathbb{R} : [x, b) \subseteq S\}$. Show that, A_x and B_x are both non-empty.
- (b) Where $x \in S$ as above, if A_x is bounded below, let $a_x = \inf(A_x)$. Otherwise, let $a_x = -\infty$, and define b_x in a corresponding manner. Show that $x \in I_x = (a_x, b_x) \subseteq S$.
- (c) Show that $S = \cup_x I_x$.
- (d) Show that the intervals I_x give a partition of S , *i.e.*, for $x, y \in S$, either $I_x = I_y$ or $I_x \cap I_y = \emptyset$.
- (e) Show that the set of distinct intervals $\{I_x : x \in S\}$ is countable.
- (f) Prove proposition 3.1.4, Pg. I-112 in Dr. Flaschka's notes.

6.3 (X, \mathcal{T}) and (Y, \mathcal{S}) are two topological spaces.

- (a) If $W \subseteq X$ is a subset, show that the *relative* topology of W defined by $O \subseteq W$ is open only if $O = W \cap U$ for some $U \in \mathcal{T}$ is indeed a topology.
- (b) If (X, d) is a metric space, and $W \subseteq X$, then we have shown in an earlier homework that the restriction of d to W gives a metric space. We can define two topologies on W , the relative topology on W from the metric topology on X , and the metric topology on W induced by the restriction of d onto W . Show that the two topologies are the same.
- (c) For every $(x, y) \in X \times Y$, define the collection of neighborhoods by

$$\mathcal{N}(x, y) = \{U_x \times V_y \mid x \in U_x \in \mathcal{T}, y \in V_y \in \mathcal{S}\}$$

Show that this collection of neighborhoods defines a *local base*. The topology generated by this local base is called the *product topology*.

6.4 (X, \mathcal{T}_1) and (X, \mathcal{T}_2) are topological spaces. The topology \mathcal{T}_1 is said to be *finer* than \mathcal{T}_2 if $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

- (a) Show that \mathcal{T}_1 is finer than \mathcal{T}_2 if and only if for every $x \in X$ and $(U \ni x) \in \mathcal{T}_2$, there is a $(V \ni x) \in \mathcal{T}_1$ such that $V \subseteq U$.
- (b) Show that the l^1 and l^2 metrics on \mathbb{R}^2 generate the same topology.

- (c) Let $X = l^1(\mathbb{R}, \mathbb{N})$. Find a set $A \subseteq X$ that is open with respect to the l^1 metric topology on X , but not open with respect to the l^∞ metric topology on X .
- (d) Let $X = C([0, 1], \mathbb{R})$ and let d_1 and d_2 be the L^1 and L^2 metrics respectively. Show that the metric topology generated by the L^2 metric is strictly finer than the metric given by the L^1 topology.