Math 527A – Fall 07
Homework 7 : Due Nov. 10

7.1 If $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ and $g : (X, \mathcal{S}) \to (Z, \mathcal{V})$ are continuous, show that the composition $g \circ f : (X, \mathcal{T}) \to (Z, \mathcal{V})$ is also continuous.

7.2 Local base for a topological space

- $\mathcal{N}(x)$ is a local base for a topological space (X, T). If $A \in \mathcal{N}(x)$, show that there exists a $U \in T$ such that $x \in U \subseteq A$.
- $\mathcal{N}_1(x)$ and $\mathcal{N}_2(x)$ are local bases for a space X. Show that the topology T_1 generated by $\mathcal{N}_1(x)$ is finer that the topology T_2 generated by $\mathcal{N}_2(x)$ if and only if for all $B \in \mathcal{N}_2(x)$, there is a set $A \in \mathcal{N}_1(x)$ such that $x \in A \subseteq B$.
- Let $X = l^\infty(\mathbb{R}, \mathbb{N})$. For each $x \in X, m \in \mathbb{N}, \epsilon > 0$, let $U(x; m, \epsilon) = \{y \in X \mid \max_{1 \leq i \leq m} i^2 |x_i - y_i| < \epsilon\}$. If $\mathcal{N}(x) = \{U(x; m, \epsilon), m \in \mathbb{N}, \epsilon > 0\}$. Show that $\mathcal{N}(x)$ is a local base, and the l^∞ metric topology on X is strictly finer than the topology T generated by this local base.
- The topology T is as defined in the previous item. Show that $x^{(n)} \in X$ converges to z in (X, T) if and only if $x_i^{(n)} \to z_i$ for all i.
- Same setup as above. Let $V(x; \epsilon) = \{y \in X \mid |x_i - y_i| < \epsilon, \forall i \in \mathbb{N}\}$. If $\mathcal{M}(x) = \{V(x; \epsilon), \epsilon > 0\}$, is $\mathcal{M}$ a local base?

7.3 Problems 3.2.7 in Prof. Flaschka's notes. We defined continuity by definition 3.2.5. If we show this is equivalent to 3.2.3, then we show that the definition of continuity is purely topological, and does not depend on the particular local base chosen.

7.4 Problems 3.2.23 and 3.2.24, Pgs. I-126, 127 in Prof. Flaschka's notes.

7.5 First Countable local base

$(X, \mathcal{T})$ is first countable, if it has a local base $\mathcal{N}(x)$ such that at every point $x \in X$, the collection of neighborhoods $\mathcal{N}(x)$ is countable.

(a) Show that the metric topology on a metric space (X, d) is first countable.

(b) Is $(X\mathcal{T})$ is first countable, show that every point $x \in X$ has a countable collection of open neighborhoods $U_n \ni x$ such that $U_n \subseteq U_{n+1}$, and x is an interior point for an open set O if and only if there is an index n such that $x \in U_n \subseteq O$.

(c) With the same defintitions as the previous part, show that if you construct a sequence by picking arbitrary points $y_n \in U_n$, it follows that the sequence $\{y_n\}$ converges.

(d) If $(X, \mathcal{T})$ is first countable, and $(Y, \mathcal{S})$ is any topological space, show that $f : X \to Y$ is continuous if and only if it is sequentially continuous. (See Problem 3.2.16 in Prof. Flaschka's notes)

7.6 A set in a topological space is *closed* if it's complement is open.

(a) If $f : X \to \mathbb{R}$ is a continuous function, show that $f^{-1}([0, 1])$ is closed in X.

(b) Problem 3.3.9, Pg. I-134 of Prof. Flaschka's notes.

(c) Problems 3.3.10 and 3.3.11, Pg. I-134, Prof. Flaschka's notes.