Math 527A – Fall 06
Homework 8 : Due Nov. 17

8.1 Local base, base and subbase

(a) Exercise 3.4.7, Pg. I-138 in Prof. Flaschka's notes. (This is the statement that we can define topologies through bases as described in class)

(b) Prove Proposition 3.4.13, Pg. I-139. (This is part of problem 3.4.12)

(c) Recall that a topological space is *second countable* if it has a countable base. Show that $\mathbb{R}$ with the usual topology is second countable.

(d) If $(X, \mathcal{T})$ is a second countable topological space, show that the topology is first countable.

(e) Problem 3.5.39, Pg. I-150 of Prof. Flaschka's notes.

(f) Show that l^∞ with the metric topology is first countable, but not second countable.

8.2 Problem 3.4.14, Pg. I-139 of Prof. Flaschka's notes.

8.3 Problem 3.4.31, Pg. I-143 of Prof. Flaschka's notes.

8.4 A function $f : \mathbb{R} \to \mathbb{R}$ is *lower semi continuous* if for all $x \in \mathbb{R}, \epsilon > 0$, there exists a $\delta > 0$ such that $|y - x| < \delta$ implies that $f(y) > f(x) - \epsilon$.

(a) Show that the collection $\mathcal{B} = \{(\alpha, \infty) \,|\, \alpha \in \mathbb{R}\}$ is a base. Let T' denote the topology generated by $\mathcal{B}$. Show that $T' \subset T_{metric}$ and the containment is strict. (Hint: One idea is to show that T' is not Hausdorff.)

(b) Show that the collection $\mathcal{B}$ along with the empty set and all of $\mathbb{R}$ is the topology generated by the base $\mathcal{B}$, *i.e.* $T' = \mathcal{B} \cup \{\emptyset, \mathbb{R}\}$.

(c) Show that T' is second countable.

(d) Show that a function $f : (\mathbb{R}, T_{metric}) \to (\mathbb{R}, T')$ is continuous, if and only if it is lower semi-continuous by the earlier definition.

(e) Show that a function $f : (\mathbb{R}, T') \to (\mathbb{R}, T_{metric})$ is continuous, if and only if it is a constant function.