Math 527A – Fall 08
Homework 9 : Due Nov. 26

9.1 Problem 3.5.16, Pg. I-149 of Prof. Flaschka's notes.

9.2 Problem 3.5.22, Pg. I-149 of Prof. Flaschka's notes.

9.3 Accumulation points

(a) Problem 3.5.26, 3.5.27, 3.5.28 and 3.5.29, Pg. I-149 of Prof. Flaschka's notes.

(b) (X, T) is a topological space and the topology T is Hausdorff. $A \subseteq X$. Let A' denote the set of accumulation points of A. Show that

$$A' = \bigcap_{x \in A} \overline{A \backslash \{x\}}.$$

9.4 Boundary points

(a) Problem 3.5.30, Pg. I-150 of Prof. Flaschka's notes.

(b) Show that the above definition of boundary point agrees with the definition in class, *viz.*, $x \in X$ is a boundary point of A iff for every neighborhood U of x, $U \cap A \neq \emptyset$ and $U \cap A^c \neq \emptyset$.

9.5 A topology on $\mathbb{N}$

Let $X = \mathbb{N} \cup \{e\}$. Define a collection $\mathcal{T}$ by $A \subseteq X$ is in T if and only if A does not contain e (this includes the empty set) or $e \in A$ and A^c is finite (this includes X).

(a) Show that T is a topology on X.

(b) Show that T is second countable.

(c) Show that $\mathbb{N}$ is dense in (X, T).

(d) A function $f : \mathbb{N} \to \mathbb{R}$ is the same thing as a sequence x_n. We will say that $g : X \to \mathbb{R}$ is a continuous extension of f if $g(n) = f(n) \, \forall n \in \mathbb{N}$. Show that f has a continuous extension iff $x_n = f(n)$ is a convergent sequence. Further, the continuous extension is given by $g(e) = \lim_{n \to \infty} f(n)$.

(e) Every element $a = (l, a_1, a_2, a_3, \ldots) \in \mathbb{R} \times \mathbb{R}^{\mathbb{N}}$ defines a function $f_a : X \to \mathbb{R}$ by $f_a(n) = a_n, f_a(e) = l$. Let $Y \subset \mathbb{R} \times \mathbb{R}^{\mathbb{N}}$ denote the set of all the convergent sequences with their associated limits, *i.e.* $(l, a_1, a_2, a_3, \ldots) \in Y \implies a_n \to l$. Find the weakest topology on X such that for all $a \in Y$, $f_a : X \to \mathbb{R}$ is continuous.

(f) Can you find a metric on X such that the metric topology is identical to the topology T above? Any topology with this property is said to be *metrizable*.