NORMED LINEAR SPACES AND DUALS

SHANKAR VENKATARAMANI

Review Dr. Flaschka's notes for the definition of linear space and norm.

1. Linear maps

Given two vector spaces V and W over a field F, a map $f : V \to W$ is a linear map if

$$f(\alpha u + \beta v) = \alpha f(u) + \beta f(v), \qquad \forall \alpha, \beta \in F, \quad u, v \in V$$

In order to define boundednedd for such linear maps, we need additional structure on the vector spaces V and W. In particular, we can define the notion of a *bounded linear map* if V and W are both normed linear spaces.

Remark 1. *A linear map $f : V \to W$ is* bounded *if $\exists C < \infty$ such that $\|f(v)\|_W \leq C\|v\|_V, \forall v \in V$.*

Note that, not every linear map is bounded if V or W is infinite dimensional, in contrast to the situation for maps between finite dimensional normed linear spaces.

For example consider the derivative operator as a linear map between $C^1([0,1],\mathbb{R})$ and $C([0,1],\mathbb{R})$ where we use the sup norm for both the vector spaces.

If $f_n(x) = \sin(n\pi x)/n$ is a sequence of functions in the domain $[0,1]$, then, $\|f_n\|_\infty = 1/n \to 0$, so that $f_n \to 0$. However, $\|f_n'\|_\infty = \pi$ for all n so that $f_n' \not\to 0$, showing that the map $f \mapsto f'$ is *unbounded*.

In class we have used the symbol $\mathcal{B}(V,W)$ to denote all bounded linear maps between V and W. This agrees with the definitions used in the book by Hunter and Nachtergaele ([**?**] henceforth).

The set $\mathcal{B}(V,W)$ inherits the structure of a linear space from W. If W is a vector space over F, and $f, g \in \mathcal{B}(V,W), \alpha, \beta \in F$, we define $\alpha f + \beta g$ to be the map $v \mapsto \alpha f(v) + \beta g(v)$ for all $v \in V$. It is easy to check that $\alpha f + \beta g$ is also a bounded linear map.

In fact, $\mathcal{B}(V,W)$ is a normed linear space where the *induced norm* is defined by

$$\|f\|_{\mathcal{B}(V,W)} = \sup_{\|v\|_V \neq 0} \frac{\|f(v)\|_W}{\|v\|_V}.$$

The fact that this operation defines a norm is proved, for instance, in Prop. 2.3.40 on Pg. I-76 of the class notes.

Remark 2. *There are other equivalent definitions of this norm. A very useful exercise is to show that*

$$\begin{aligned}\sup_{\|v\|_V \neq 0} \frac{\|f(v)\|_W}{\|v\|_V} &= \inf\{C \in \mathbb{R} \mid \quad \|f(v)\|_W \leq C\|v\|_V \quad \forall v \in V\} \\ &= \sup_{\|v\|_V = 1} \|f(v)\|_W \\ &= \sup_{\|v\|_V \leq 1} \|f(v)\|_W.\end{aligned}$$

A particular case of this general construction is of prticular interest, the case where $W = \mathbb{R}$. Assume that V is a normed linear space over $\mathbb{R}$. Then the space $V^* = \mathcal{B}(V, \mathbb{R})$ is called the dual of V, and it consists of all bounded, real valued, linear functionals on V.

For example, if V is $C([0,1], \mathbb{R})$ with the sup norm, then the maps $f \mapsto f(0.5)$ and $f \mapsto \int_0^1 x^2 f(x)dx$ are indeed elements of the dual space V^*. However if V is the linear space $C([0,1], \mathbb{R})$ with the L^1 norm, then the map $f \mapsto f(0.5)$ is *linear but unbounded*, and hence does not belong to the dual V^*.

2. SEMINORMS

V is a vector space over reals. A seminorm is a non-negative real valued function $p : V \to [0, \infty)$, that satisfies all the requirements on a norm, except the condition $p(v) = 0 \implies v = 0$. In particular, p needs to satisfy the triangle inequality.

Remark 3. *Note that, every norm is also a seminorm.*

A common scenario in which seminorm's arise is given by the following proposition.

Proposition 1. *If $L : V \to \mathbb{R}$ is a linear map, then the function defined by $p(v) = |L(v)|$ is a seminorm.*

Proof. Clearly $p(v) \geq 0$ for all $v \in V$. Also, $p(\alpha v) = |L(\alpha v)| = |\alpha L(v)| = |\alpha| p(v)$. Finally, $p(v_1 + v_2) = |L(v_1 + v_2)| = |L(v_1) + L(v_2)| \leq |L(v_1)| + |L(v_2)| = p(v_1) + p(v_2)$.

□

Remark 4. *Note that we don't need a map into $\mathbb{R}$. If $L : V \to W$ is a linear map, and ρ is a* seminorm *on w, then $p(v) = \rho(L(v))$ is also a seminorm, and this fact is proved in exactly the same manner as above.*

Example 1. *Show that,*

$$p(f) = \sqrt{\left|\int_0^1 f(t)dt\right|^2 + [f(0.5)]^2}$$

is a seminorm on $C([0,1])$.

We can define a linear map from $C([0,1])$ to $\mathbb{R}^2$ by

$$f \mapsto L(f) = \left(\int_0^1 f(t)dt, f(0.5)\right).$$

Consequently, $\|L(f)\|_2$ is a seminorm.

Proposition 2. *V is a vector space over a field F. Given a family of seminorms $p_\alpha, \alpha \in A$, it follows that*

$$p(v) = \sup_{\alpha \in A} p_\alpha(v)$$

is a seminorm on a vector space V' defined by

$$V' = \{v \in V \mid \sup_{\alpha \in A} p_\alpha(v) < \infty\}$$

Remark 5. *You've proved this in your homework.*

Remark 6. *The proposition has two distinct aspects, which require proof. Firstly, we are constructing a space V', which we must show is indeed a vector space. After that, we need to show that the function p, which is clearly well defined on V' is a seminorm. The proof for the second part has exactly the same ideas as the proof in the class notes that the induced norm is indeed a norm.*

Proof. We first show that V' is indeed a vector space.

Assume that $v_1, v_2 \in V'$ and c_1, c_2 are scalars in the underlying field F. Then $\alpha v_1 + \beta v_2 \in V$, because $V' \subseteq V$ and V is a vector space.

For each $\alpha \in A$, we have

$$\begin{aligned} p_\alpha(c_1 v_1 + c_2 v_2) \leq & p_\alpha(c_1 v_1) + p_\alpha(c_2 v_2) \\ = & |c_1| p_\alpha(v_1) + |c_2| p_\alpha(v_2) \\ \leq & |c_1| \sup_{\beta \in A} [p_\beta(v_1)] + |c_2| \sup_{\gamma \in A} [p_\gamma(v_2)] \\ = & |c_1| p(v_1) + |c_2| p(v_2) < \infty \end{aligned}$$

Since this is true for each $\alpha \in A$, we see that

$$\sup_{\alpha \in A} p_\alpha(c_1 v_1 + c_2 v_2) \leq |c_1| p(v_1) + |c_2| p(v_2) < \infty$$

implying that $c_1 v_1 + c_2 v_2 \in V'$. This proves that V' is a vector space.

Also, the last equation implies that

$$p(c_1 v_1 + c_2 v_2) \leq |c_1| p(v_1) + |c_2| p(v_2)$$

for all $v_1, v_2 \in V', c_1, c_2 \in F$. Taking $c_1 = c_2 = 1$ yields the triangle inequality. The other requirements for p to be a seminorm are easily verified, and this proves that p is indeed a seminorm. □

A natural question at this stage is under what circumstances is p a norm? This is answered by the following proposition

Proposition 3. *Using the same notation as above, p is a norm on V' if, for every $v \neq 0$ in V, there is an $\alpha \in A$ such that $p_\alpha(v) > 0$.*

The proof is straightforward, since, if $v \neq 0$ and $v \in V' \subseteq V$, we see that

$$p(v) \geq p_\alpha(v)$$

for all $\alpha \in A$, implying that $p(v) > 0$, so that $p(z) = 0 \implies z = 0$.

The following examples illustrate this strategy for constructing norms.

Example 2. *Let $L_i(\mathbf{x}) = x_i$ for $i = 1, 2, \ldots, n$ be n linear functions from $\mathbb{R}^n$ to $\mathbb{R}$ such that L_i extracts the ith component of a point $\mathbf{x} \in \mathbb{R}^n$.*

By an earlier result, the functions $p_i(\mathbf{x}) = |x_i|$ are seminorms. Also, the function $\sup_{i \in \{1,2,\ldots,n\}} |x_i|$ is a norm since for every $\mathbf{x} \neq 0$ in $\mathbb{R}^n$, at least one of its components is non zero, which means at least one of the p_i is positive.

Example 3. *Let $V = M_{2\times 2}(\mathbb{R})$ be the set of all 2×2 matrices with real entries. For all $\mathbf{x} = (a, b)^T \in \mathbb{R}^2$, we can define a linear map from V to $\mathbb{R}^2$ by*

$$\mathbf{Q} \in V \mapsto \mathbf{Q}\mathbf{x} \in \mathbb{R}^2.$$

Consequently, it follows that, for each $\mathbf{x} \in \mathbb{R}^2$, the mapping $p_{\mathbf{x}}(\mathbf{Q}) = \|\mathbf{Q}\mathbf{x}\|_2$ is a seminorm.

For instance, if we take $\mathbf{x} = (1, -1)^T$, then the above construction yields the mapping

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \|(a-b, c-d)\|_2 = \sqrt{(a-b)^2 + (c-d)^2}$$

which is indeed a seminorm. Note that, this is not a norm because the matrix

$$\mathbf{Q} = \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}$$

is nonzero, but $p_{(1,-1)}(\mathbf{Q}) = 0$.

If we consider the set of all $\mathbf{x} \in \mathbb{R}^2$, it is easy to see that

$$\sup_{\mathbf{x} \in \mathbb{R}^2} \|\mathbf{Q}\mathbf{x}\|_2 = \infty$$

for all $\mathbf{Q} \neq 0$. Thus it doesn't yield a useful norm on the space of matrices.

However, if we restrict ourselves to the set $\{\mathbf{x} \in \mathbb{R}^2 \,|\, \|\mathbf{x}\|_2 = 1\}$, we see that

$$p(\mathbf{Q}) = \sup_{\|\mathbf{x}\|_2 = 1} p_{\mathbf{x}}(\mathbf{Q}) = \sup_{\|\mathbf{x}\|_2 = 1} \|\mathbf{Q}\mathbf{x}\|_2$$

is indeed a seminorm that is defined on all of $M_{2\times 2}$.

This is also a norm, because $\mathbf{Q} \neq 0$ implies either $\mathbf{Q}(0,1)^T \neq 0$ or $\mathbf{Q}(1,0)^T \neq 0$, whence it follows that $p(\mathbf{Q}) > 0$.

Note that, this norm p is precisely the induced norm on the space of 2×2 matrices considered as a space of linear maps between $\mathbb{R}^2$ with the Euclidean norm and itself.

3. Induced norms and duality

As before, we define $\mathcal{B}(V, W)$ as the space of all "bounded" linear maps between V and W with respect to the induced norm

$$\|f\|_{\mathcal{B}(V,W)} = \sup_{\|v\|_V \neq 0} \frac{\|f(v)\|_W}{\|v\|_V}.$$

In light of the above discussion, we can also interpret the induced norm as follows:

Every element $v \in V$ is a linear map from $\mathcal{L}(V, W)$ to W by

$$f \in \mathcal{L}(V, W) \mapsto f(v) \in W$$

Consequently, the mapping $p_v : \mathcal{L}(V, W) \to [0, \infty)$ defined by $p_v(f) = \|f(v)\|_W$ is a seminorm on $\mathcal{L}(V, W)$.

We can now construct a seminorm by taking the supremum of p_v over an appropriate set of vectors v. If we take the supremum over all $v \in V$, it is easy to see that the supremum is infinity unless $f = 0$, so this is not a useful construction.

Another idea is to take the supremum only over those $v \in V$ with $\|v\|_V = 1$. This will define a seminorm, but not necessarily on $\mathcal{L}(V, W)$. Rather, the seminorm is only defined on the set

$$\{f \in \mathcal{L}(V, W) \,|\, \sup_{\|v\|=1} \|f(v)\|_W < \infty\}.$$

This is precisely the space that we denoted by $\mathcal{B}(V, W)$, that is, all linear maps with the property that there is a $C < \infty$ such that

$$\|f(v)\|_W \leq C \quad \forall v \in V \text{ such that } \|v\|_V = 1.$$

This justifies, among other things, why $\mathcal{B}(V,W)$ is itself a linear vector space, and is the set on which the induced norm was originally defined. Note that, the seminorm defined by

$$p(f) = \sup_{\|v\|=1} \|f(v)\|_W$$

is indeed a norm, since $f(v) = 0$ for all v implies that $f = 0$, that is for every $f \neq 0$, there is a v such that $f(v) \neq 0$, and setting $u = v/\|v\|$ gives an element u with $\|u\| = 1$ such that $f(u) \neq 0$.

The particular case where the range space of the linear maps is $\mathbb{R}$ gives the *dual* of V, that is

$$V^* = \mathcal{B}(V, \mathbb{R})$$

Given a concrete space V, the problem of finding the dual V^* is often solved in two steps, first finding the appropriate set of linear maps that define V^* and then finding the induced norm for the dual.

I will illustrate this below with a few examples:

Example 4. *$V = \mathbb{R}^n$. Find all the linear maps $\phi : V \to \mathbb{R}$.*

Remark 7. *Note that, we haven't defined any norm on V! The "answer" to this question is seemingly independent of the norm that we choose to impose on V.*

Going back to the question posed in the example, let $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_n}$ be the standard basis for $\mathbb{R}^n$. Then, for any $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we have $\mathbf{x} = x_1\mathbf{e_1} + x_2\mathbf{e_2} + \ldots + x_n\mathbf{e_n}$. Consequently, if $f : \mathbb{R}^n \to \mathbb{R}$ is a linear function, we have

$$f(\mathbf{x}) = x_1 f(\mathbf{e_1}) + x_2 f(\mathbf{e_2}) + \ldots + x_n f(\mathbf{e_n}) = a_1x_1 + a_2x_2 + \ldots a_nx_n = \mathbf{a}\cdot\mathbf{x}$$

where $a_i = f(\mathbf{e_i})$ and $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$.

Conversely, given any point $\mathbf{b} \in \mathbb{R}^n$, it is easy to see that

$$g(\mathbf{x}) = \mathbf{b}\cdot\mathbf{x}$$

does indeed define a linear function $g : \mathbb{R}^n \to \mathbb{R}$.

Consequently, we see that the set of linear functions from $\mathbb{R}^n$ to $\mathbb{R}$ is in a one to one correspondence with $\mathbb{R}^n$ and we can indeed represent all linear functionals on $\mathbb{R}^n$ as $\mathbf{x} \mapsto \mathbf{a}\cdot\mathbf{x}$ for some $\mathbf{a} \in \mathbb{R}^n$.

As discussed in class, an alternative (and sometimes useful) way to think of linear functions is in terms of it's level sets. If $f : \mathbb{R}^n \to \mathbb{R}$ is a non-trivial function, *i.e.* not identically zero, then the set $f^{-1}(0)$, *i.e* the kernel of the linear map f defines a $n-1$ dimensional hypersurface in $\mathbb{R}^n$. Likewise, the family of hypersurfaces $f^{-1}(c), c \in \mathbb{R}$ gives a foliation of $\mathbb{R}^n$ by parallel hypersurfaces, that correspond to the level sets of the function f.

Given this family of hypersurfaces, it is of course easy to recover the corresponding map $x \mapsto \mathbf{a}\cdot\mathbf{x}$ using the results $a_i = f(\mathbf{e_i}), i = 1, 2, \ldots, n$.

Remark 8. *We have addressed the question of the* representation of all linear functions *on a vector space. For $V = \mathbb{R}^n$, it is easy to answer this qiestion, but for other (infinite-dimensional) spaces, the answers can be surprisingly complicated.*

Example 5. *Find the dual of the $(\mathbb{R}^n, l^p)$ with $1 \leq p \leq \infty$.*

By the preceding argument, if f is a linear map from $\mathbb{R}^n$ to $\mathbb{R}$, there is a vector $\mathbf{a} \in \mathbb{R}^n$ such that

$$f(\mathbf{v}) = \mathbf{a} \cdot \mathbf{v} = \sum_{i=1}^{n} a_i v_i$$

for all $\mathbf{v} \in \mathbb{R}^n$.

Conversely, every $\mathbf{a} \in \mathbb{R}^n$ does induce a bounded linear map from $(\mathbb{R}^n, l^p)$ to $\mathbb{R}$ by $\mathbf{x} \mapsto \mathbf{a} \cdot \mathbf{x}$ since Hölder's inequality implies that

$$|\mathbf{a} \cdot \mathbf{x}| \leq \|\mathbf{x}\|_p \|\mathbf{a}\|_q$$

where $p^{-1} + q^{-1} = 1$.

We have thus identified the set of linear maps in the dual. Now we have to calculate the induced norm.

For $f(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$,

$$\frac{|\mathbf{a} \cdot \mathbf{x}|}{\|\mathbf{x}\|_p} \leq \|\mathbf{a}\|_q$$

so that $\|f\|_{induced} \leq \|\mathbf{a}\|_q$. Also, given $\mathbf{a} \neq 0 \in \mathbb{R}^n$ set

$$x_i = \begin{cases} \text{sign}(a_i) & p = 1, |a_i| = \|\mathbf{a}\|_\infty \\ 0 & p = 1, |a_i| < \|\mathbf{a}\|_\infty \\ \text{sign}(a_i)|a_i|^{p/(p-1)} & 1 < p < \infty \\ \text{sign}(a_i) & p = \infty \end{cases}$$

In each of these cases, it is easy to check that, with these choices

$$\frac{|\mathbf{a} \cdot \mathbf{x}|}{\|\mathbf{x}\|_p} = \|\mathbf{a}\|_q,$$

implying that $\|f\|_{induced} \geq \|\mathbf{a}\|_q$ (CHECK THESE CLAIMS!).

Combining the two inequalities, we see that the induced norm is the l^q norm, so that the dual space is $V^* = (\mathbb{R}^n, l^q)$ with $q^{-1} = 1 - p^{-1}$.

Example 6. *If $V = \mathbb{R}^3$ with a norm $\|(x_1, x_2, x_3)\|_a = |x_1| + \sqrt{x_2^2 + x_3^2}$, determine the dual V^*.*

The same argument as above shows that, as a set, $V^* = \mathbb{R}^3$. we need to calculate the induced norm on $\mathbb{R}^3$ to determine the dual.

The unit ball in $\|\cdot\|_a$ is given by the set

$$|x_1| + \sqrt{x_2^2 + x_3^2} = 1$$

For $(a_1, a_2, a_3) \in \mathbb{R}^3$, the induced norm is

$$\begin{aligned} \|(a_1, a_2, a_3)\|_{V^*} &= \sup_{|x_1| + \sqrt{x_2^2 + x_3^2} = 1} |a_1 x_1 + a_2 x_2 + a_3 x_3| \\ &\leq \sup_{|x_1| + \sqrt{x_2^2 + x_3^2} = 1} |a_1 x_1 + \sqrt{a_2^2 + a_3^2}\sqrt{x_2^2 + x_3^2}| \\ &\leq \max(|a_1|, \sqrt{a_1^2 + a_2^2}) \end{aligned}$$

where we have used the Cauchy-Schwarz inequality in the second line, and $|x_1|+\sqrt{x_2^2+x_3^2} \leq 1$ in obtaining the last line.

This suggests that the norm on the dual is

$$\|(a_1, a_2, a_3)\|_{induced} = \max(|a_1|, \sqrt{a_2^2+a_3^2})$$

To show that this is indeed the case, we need to show there is an $\mathbf{x} \in \mathbb{R}^3$ such that

$$|a_1 x_1 + a_2 x_2 + a_3 x_3| \geq \max\left(|a_1|, \sqrt{a_2^2+a_3^2}\right) \|\mathbf{x}\|_a$$

From the above analysis, it is clear that to get an equality in the 3rd line, we need $|x_1| = 1$ if $|a_1| \geq \sqrt{a_2^2+a_3^2}$ and $|x_1| = 0$ otherwise. Also, for equality in the Cauchy-Schwarz inequality, we need $x_2 = a_2, x_3 = a_3$. Combining all of these requirements, we see that, for the choice

$$x_1 = \begin{cases} \mathrm{sign}(a_1) & |a_1| \geq \sqrt{|a_2|^2+|a_3|^2} \\ 0 & \text{otherwise} \end{cases}$$

$$x_2 = \begin{cases} 0 & |a_1| \geq \sqrt{|a_2|^2+|a_3|^2} \\ a_2 & \text{otherwise} \end{cases}$$

$$x_3 = \begin{cases} 0 & |a_1| \geq \sqrt{|a_2|^2+|a_3|^2} \\ a_3 & \text{otherwise} \end{cases}$$

we see that

$$|a_1 x_1 + a_2 x_2 + a_3 x_3| = \max\left(|a_1|, \sqrt{a_2^2+a_3^2}\right) (|x_1| + \sqrt{x_2^2+x_3^2})$$

Combining this with the preceding inequality, we obtain

$$\|(a_1, a_2, a_3)\|_{induced} = \max\left(|a_1|, \sqrt{a_2^2+a_3^2}\right)$$

Example 7. *Define $p : \mathbb{R}^2 \to [0, \infty)$ by*

$$p(x, y) = \begin{cases} |x| + |y| & xy \geq 0 \\ \sqrt{x^2+y^2} & \textit{otherwise} \end{cases}$$

(a) Show that p is norm on $\mathbb{R}^2$.

(b) Determine the dual space (State clearly any identifications you make between spaces of functions and points in $\mathbb{R}^n$.)

We will construct the norm out of the linear functions defining the tangents to the unit ball.

If (x, y) is in the first or the third quadrant and $p(x, y) = 1$, it is clear that $x+y = \pm 1$. Further, the lines $x + y = \pm 1$ are tangent to the unit ball.

If (x, y) is in the second or the fourth quadrant, and $p(x, y) = 1$, it follows that $x^2+y^2 = 1$ so $x = \pm\cos(\theta), y = \pm\sin(\theta)$ for $\theta \in [\pi/2, \pi]$ and the tangents to the unit ball at the point $(\cos(\theta), \sin(\theta))$ is given by $x\cos(\theta) + y\sin(\theta) = 1$.

Based on this intuition, we claim:

$$p(x, y) = \max(|x + y|, \sup_{\theta\in[\pi/2,\pi]} |x\cos(\theta) + y\sin(\theta)|)$$

For all x, y, it is clear that $|x\cos(\theta) + y\sin(\theta)| \leq \sqrt{x^2+y^2}$ by the Cauchy-Schwarz inequality.

If x and y have the same sign, we have

$$|x+y|^2 = x^2 + 2xy + y^2 \geq x^2 + y^2$$

so that, if $xy > 0$, it follows that

$$\max(|x+y|, \sup_{\theta\in[\pi/2,\pi]} |x\cos(\theta) + y\sin(\theta)|) = |x+y| = p(x,y)$$

If $xy \leq 0$, setting $\theta_0 = \arctan(y/x)$ where the arctangent is defined on a fundamental domain $(0, \pi]$, we see that $y/x \leq 0$ so that $\theta_0 \in [\pi/2, \pi]$. Therefore

$$\sup_{\theta\in[\pi/2,\pi]} |x\cos(\theta) + y\sin(\theta)| \geq |x\cos(\theta_0) + y\sin(\theta_0)| = \sqrt{x^2+y^2}$$

Combining this with the result above from the Cauchy-Schwarz inequality, we see that

$$sup_{\theta\in[\pi/2,\pi]} |x\cos(\theta) + y\sin(\theta)| = \sqrt{x^2+y^2}.$$

Since $xy \leq 0$, we also have

$$|x+y|^2 = x^2 + 2xy + y^2 \leq x^2 + y^2$$

so that, if $xy \leq 0$, it follows that

$$\max(|x+y|, \sup_{\theta\in[\pi/2,\pi]} |x\cos(\theta) + y\sin(\theta)|) = \sqrt{x^2+y^2} = p(x,y)$$

Thus, we have shown that

$$p(x,y) = \sup_{\alpha} |L_\alpha(x,y)|$$

where L_α is a suitable family of linear functions, in particular the linear functions that determine the tangents to the unit ball.

By Proposition 2 it follows that p is a seminorm. Also, if $(x,y) \neq (0,0)$ it follows from the definition that $p(x,y) \neq 0$. This proves that p is a norm.

We will now compute the dual space. As before, there is a one to one identification between V^* and $\mathbb{R}^2$ by for all $\mathbf{a} \in \mathbb{R}^2$, $f(x) = \mathbf{a}\cdot\mathbf{x}$ and for $i = 1, 2$ $a_i = f(\mathbf{e}_i)$.

Also, if p^* is the induced norm on the dual, for $(a_1, a_2) \in \mathbb{R}^2$, the induced norm is

$$\begin{aligned} p^*(a_1,a_2) &= \sup_{p(x,y)=1} |a_1x + a_2y| \\ &\leq \min\left[\sqrt{a_1^2+a_2^2} \sup_{p(x,y)=1} \sqrt{x^2+y^2}, \max(|a_1|,|a_2|) \cdot \sup_{p(x,y)=1} (|x|+|y|)\right] \end{aligned}$$

where we have used the Hölder's inequality twice in the second line, and recognized that $r \leq s, r \leq t \implies r \leq \min(s,t)$.

If $p(x,y) = 1$ either $\sqrt{x^2+y^2} = 1$ or $xy > 0 \implies \sqrt{x^2+y^2} = \sqrt{1-2xy} \leq 1$. Thus, we get

$$p^*(a_1,a_2) \leq \sqrt{a_1^2+a_2^2}$$

Without loss of generality, $(a_1, a_2) \neq (0,0)$. If $a_1a_2 \leq 0$ setting

$$x = \frac{a_1}{\sqrt{a_1^2+a_2^2}}, \quad y = \frac{a_2}{\sqrt{a_1^2+a_2^2}},$$

we get $xy \le 0$ and $p(x, y) = 1$ and

$$|a_1 x + a_2 y| = \sqrt{a_1^2 + a_2^2}.$$

Therefore for $a_1 a_2 \le 0$ we get $p^*(a_1, a_2) = \sqrt{a_1^2 + a_2^2}$.

If $a_1 a_2 > 0$, we have $p(x, y) = 1 \implies 0 \le x \le 1$ and $0 \le y \le 1$. If $xy \le 0$ but $a_1 a_2 > 0$, it follows that $a_1 x$ and $a_2 y$ do not have the same sign, so that

$$|a_1 x + a_2 y| \le \max(|a_1 x|, |a_2 y|) \le \max(|a_1|, |a_2|).$$

If $xy \ge 0$, we have $a_1 x$ and $a_2 y$ have that $p(x, y) = 1 \implies |x| + |y| = 1$, and by Hölder's inequality, we have

$$|a_1 x + a_2 y| \le \max(|a_1|, |a_2|)(|x| + |y|) \le \max(|a_1|, |a_2|).$$

Finally, considering $x = 1, y = 0$ and $x = 0, y = 1$, we see that

$$p^*(a_1, a_2) \ge \max(|a_1|, |a_2|).$$

Combining all these arguments, we see that

$$p^*(a_1, a_2) = \begin{cases} \max(|a_1|, |a_2|) & a_1 a_2 > 0 \\ \sqrt{a_1^2 + a_2^2} & \text{otherwise} \end{cases}$$

4. Infinite dimensional spaces

A key ingredient in all our computations for duals of finite dimensional spaces is the representation result that the set of all linear functions on $V = \mathbb{R}^n$ is in a one to one correspondence with $\mathbb{R}^n$ where the correspondence is

$$f \in V^* \mapsto \mathbf{a} \in \mathbb{R}^n \quad a_i = f(\mathbf{e}_i), i = 1, 2, \ldots, n$$

But not all linear functionals on infinite dimensional spaces can be represented this way.

In what follows $\mathbf{e}_i$ is the sequece that is 1 in the i–th index, but is zero otherwise.

Example 8. *c is the space of all sequences that converge. Show that $\phi(x) = \lim_{n\to\infty} x_n$ defines a bounded linear map on $(c, \|\cdot\|_\infty)$.*

Clearly, c is a linear space and ϕ is a linear map(VERIFY THIS!). Also,

$$|\phi(x) - \phi(y)| = \lim_{n\to\infty} |x_n - y_n| \le \sup_n |x_n - y_n| = \|x - y\|_\infty$$

showing that ϕ is indeed a bounded linear map.

Note also, that $\phi(\mathbf{e}_i) = 0$ for all i, showing that there is no sequence $\mathbf{a} \in \mathbb{R}^{\mathbb{N}}$ such that $\phi(x) = \sum_i a_i x_i$.

So, when can we assume that a linear function on a sequence space can be represented as $\phi(x) = \sum a_i x_i$? If this representation holds, it is clear that $a_i = \phi(\mathbf{e}_i)$.

Example 9. *$V = l_c$, the space of all compact sequences,* i.e., *sequences that are eventually zero. If $\phi : V \to \mathbb{R}$ is a linear map, then $\phi(x) = \sum_i a_i x_i$ with $a_i = \phi(\mathbf{e}_i)$.*

If $x \in l_c$, then $x = \sum_{i=1}^N x_i \mathbf{e}_i$ is a finite linear combination. Consequently,

$$\phi(x) = \phi\left(\sum_{i=1}^N x_i \mathbf{e}_i\right) = \sum_{i=1}^N a_i x_i = \sum_{i=1}^\infty a_i x_i$$

where the last equality follows from the fact that the infinite series has only finitely many non-zero terms, so it converges.

Note that, there are no *apriori* restrictions on the coefficients a_i so the set of all linear functionals on l_c is in a one to one correspondence with $\mathbb{R}^{\mathbb{N}}$, and this is independent of any norm that we choose to impose on l_c.

Example 10. *Let* $V = (l_c, \|.\|_p)$. *Find* V^*.

From the representation result in the previous example, we see that $\phi \in V^*$ implies that

$$\phi(x) = \sum_i a_i x_i$$

so that

$$|\phi(x)| \leq \|a\|_q \|x\|_p$$

Consequently, if $\|a\|_q < \infty$, the corresponding linear function ϕ is a bounded linear map. Also, the standard argument taking

$$x_i^{(N)} = \begin{cases} \frac{\text{sign}(a_i)|a_i|^{q-1}}{[\sum_{i=1}^N |a_i|^q]^{1/p}} & i \leq N \\ 0 & \text{otherwise} \end{cases}$$

along with $|\phi(x)| \leq C\|x\|_p$ implies that

$$\sum_{i=1}^N |a_i|^q \leq C^q$$

for all N. Consequently, $a \in l^q$ and the induced norm is $\|\phi\|_{\text{induced}} = \|a\|_q$.

We can use this result, along with the fact that every bounded linear functional is continuous to compute the duals of the l^p sequence spaces.

For every $y \in l^p$, it is clear that the sequence $y^{(N)}$ defined by

$$y_i^{(N)} = \begin{cases} y_i & i \leq N \\ 0 & \text{otherwise} \end{cases}$$

Clearly, $y^{(N)} \in l_c$ and $\|y^{(N)} - y\|_p \to 0$. This implies that to define a bounded linear functional on l^p it is enough to define it on l^c and then extend the function to l^p by continuity.

In particular, every *bounded* linear function on l^p can be written as

$$\phi(x) = \sum_i a_i x_i$$

where $a \in l^q$, guaranteeing that the sum converges, and the same argument as in the previous example showing that the dual space is $(l^q, \|\cdot\|_q)$ with the correspondence $\phi \mapsto \{a_i\} = \{\phi(\mathbf{e}_i\}$.

The same argument also works for the sequence space c_0 to show that the dual is l^1 (VERIFY THIS!)

This argument *will not* work for c! The sequence $\beta_0 = \{1, 1, 1, 1, \ldots\} \in c$. For any $y \in l_c$, we see that $\|y - \beta_0\|_\infty \geq 1$, since for sufficiently large values of the index, $y_n = 0$. So, there is no sequence in l_c that converges to β_0 in the sup norm.

Example 11. *Compute the dual of* $V = (c, \|\cdot\|_\infty)$.

The issue here is to find a representation of all linear functions on c. Observe, if $x \in c$, then $x - \lim_{n\to\infty} x_n \beta_0 \in c_0$, where as above $\beta_0 = \{1, 1, 1, 1, \ldots\}$. As we argued earlier, $\psi_0(x) = \lim_{n\to\infty} x_n$ is a bounded linear function on V. Consequently, if ϕ is a bounded linear functional on V, then

$$\phi(x) = \phi(\psi_0(x)\beta_0 + (x - \psi_0(x)\beta_0)) = m\psi_0(x) + \phi(x - \psi_0(x)\beta_0)$$

where $m = \phi(\beta_0)$. Since $x - \psi_0(x)\beta_0 \in c_0$, it follows that there is a sequence $x_i' = x_i - \psi_0(x)$ such that

$$\lim_{N\to\infty} \|x - \psi_0(x)\beta_0 - \sum_{i=1}^{N} x_i' \mathbf{e}_i\|_\infty = 0.$$

Consequentl, if ϕ is a bounded linear function.

$$\phi(x) = m\psi_0(x) + \lim_{N\to\infty} \sum_{i=1}^{N} a_i x_i'$$

where $a_i = \phi(\mathbf{e}_i)$.

If $a \in l^1$, by Hölders inequality, we have

$$|\phi(x)| \leq |m|\|x\|_\infty + 2\|a\|_1\|x\|_\infty$$

using the estimates

$$\left|\lim_{n\to\infty} x_n\right| \leq \|x\|_\infty, \quad \|x'\| = \|x - \psi_0(x)\beta_0\|_\infty \leq \|x|_\infty + |\psi_0(x)|\|\beta_0\|_\infty = 2\|x\|_\infty$$

showing that ϕ is a bounded function if $|m| + 2\|a\|_1 < \infty$. Also, considering the sequences β_0 and

$$x_i^{(N)} = \begin{cases} \text{sign}(a_i) & i \leq n \\ 0 & \text{otherwise} \end{cases},$$

it is easy to see that ϕ is a bounded function only if $|m| < \infty, \|a\|_i < \infty$.

We have thus identifies the space of bounded linear functionals as $\mathbb{R} \times l^1$. Computing the dual norm is tricky (we don't have equality in the upper and lower bounds above), but from the argument above, it is clear that the dual norm of $(m, a) \in \mathbb{R} \times l^1$ is equivalent to

$$\|(m, a)\| \equiv |m| + \|a\|_1 \equiv \max(|m|, \|a\|_1)$$

.