

**Exam 2**  
**Math 527a – Principles of Analysis**

All questions carry equal weight

(a)  $(X, T)$  is a topological space.  $A_i, i = 1, 2, \dots$  is a family of subsets of  $X$ . Show that  $\overline{A_1 \cup A_2} = \overline{A_1} \cup \overline{A_2}$ . Does this also extend to countable unions, *i.e.*, prove or give a counterexample to  $\overline{\cup_i A_i} = \cup_i \overline{A_i}$ .

**Solution**

Since  $A \subseteq \overline{A}$  for all sets  $A$ , we see that  $A_n \subseteq \cup_i A_i \subseteq \overline{\cup_i A_i}$  for finite, countable or even arbitrary unions. Also,  $\overline{A_n}$  is the intersection of all the closed sets containing  $A_n$ , and  $\overline{\cup_i A_i}$  is one such set. Therefore  $\overline{A_1} \subseteq \overline{A_1 \cup A_2}$ , and similarly  $\overline{A_2} \subseteq \overline{A_1 \cup A_2}$ . Taken together, we obtain

$$\overline{A_1} \cup \overline{A_2} \subseteq \overline{A_1 \cup A_2}.$$

One way of showing the converse is through the contrapositive, as we discussed in class, *i.e.* show that  $x \notin \overline{A_1} \cup \overline{A_2}$  implies  $x \notin \overline{A_1 \cup A_2}$ . Here is an alternate argument.  $A_1 \subseteq \overline{A_1}$  and  $A_2 \subseteq \overline{A_2}$ . Therefore,  $A_1 \cup A_2 \subseteq \overline{A_1} \cup \overline{A_2}$ , which is closed, so that  $\overline{A_1 \cup A_2} \subseteq \overline{\overline{A_1} \cup \overline{A_2}}$ .

The preceding argument *does not extend* to countable or arbitrary unions, and a way to make it fail is to have closed sets whose countable union is no longer closed. So, one counterexample is to take  $A_n = [\frac{1}{n}, 1]$ . Each  $A_n$  is closed, and  $\cup_n A_n = (0, 1]$ . Then, we see that

$$[0, 1] = \overline{\bigcup_{n=1}^{\infty} A_n} \neq \bigcup_{n=1}^{\infty} \overline{A_n} = (0, 1].$$

(b) Let  $X = \mathbb{R}^{\mathbb{N}}$  be the space of all real sequences and  $\mathcal{T}$  be the weakest topology such that the projections  $\pi_i : X \rightarrow \mathbb{R}$  defined by  $\pi_i(x) = x_i$  are continuous, where  $\mathbb{R}$  is endowed with the standard metric topology. Show that  $\mathcal{T}$  is second countable.

**Solution**

Since  $\mathbb{R}$  with the standard metric topology is second countable, it has a second countable base  $\mathcal{B}_M$ . For instance, we can choose  $\mathcal{B}_M$  as the collection of all open intervals  $I_k = (q_k - 1/n_k, q_k + 1/n_k)$  with rational center  $q_k$  and radius  $1/n_k, k \in \mathbb{N}$ .

Define a collection  $\mathcal{B}_W$  as consisting of all finite intersections of the form  $V = \bigcap_{j=1}^n \pi_j^{-1}(I_{k_j})$ . Since the  $\pi_j$  are continuous with respect to the weak topology, each set  $V \in \mathcal{B}_W$  is open in the weak topology. Also, the collection  $\mathcal{B}_W$  is countable since, for each  $n$ , the sets  $V$  are in one to one correspondence with sets of  $n$  natural numbers (the indices  $k_j$  of the intervals), and are hence countable, and the union over all  $n \in \mathbb{N}$  is a countable union of countable sets, and is hence countable.

We will first show that  $\mathcal{B}_W$  is a base. If  $x \in X$ , the  $\pi_1(x) = x_1 \in \mathbb{R}$ , so there is some interval  $(q-1, q+1)$  with  $q$  rational such that  $x_1 \in (q-1, q+1)$ . Thus, there exists  $V = \pi_1^{-1}((q-1, q+1)) \in \mathcal{B}_W$  such that  $x \in V$ . Since the  $x$  we started with was arbitrary, it is clear that  $\bigcup_{V \in \mathcal{B}_W} V = X$ .

Also, by construction, if  $V_1, V_2 \in \mathcal{B}_W$  and  $x \in V_1 \cap V_2$ , then there is a finite number of intervals  $I_1, I_2, I_3, \dots, I_m$  such that  $\pi_j(x) \in I_j$  for  $j = 1, 2, \dots, m$  and  $\pi_j(y) \in I_j$  for  $j = 1, 2, \dots, m \implies y \in V_1 \cap V_2$ . Since  $V_3 = \bigcap_{j=1}^m \pi_j^{-1}(I_j) \in \mathcal{B}_W$ , it follows that  $\mathcal{B}_W$  is a base.

Since every set in  $\mathcal{B}_W$  is open in the weak topology, it follows that the weak topology is stronger than the topology generated by the base  $\mathcal{B}_W$ . Conversely, a basis for the weak topology is given by sets of the form  $W = \bigcap_{m=1}^l \pi_{j_m}^{-1}(U_m)$ , where the sets  $U_m$  are open in  $\mathbb{R}$ . If  $x \in W$ , it follows that  $\pi_{j_m}(x) = x_{j_m} \in U_m$  for  $m = 1, 2, \dots, l$ . Since  $\mathcal{B}_M$  is a base for the topology on  $\mathbb{R}$  there are intervals  $I_{j_m} \in \mathcal{B}_M, m = 1, 2, \dots, m$  such that  $x_{j_m} \in I_{j_m} \subseteq U_{j_m}$ . Let  $N$  be any index larger than all the  $j_m, m = 1, 2, \dots, m$ . If  $k \leq N$  and  $k \neq j_m$  for any  $m = 1, 2, \dots, m$ , let  $I_k \in \mathcal{B}_M$  be any interval containing  $x_k$ . If we let  $V = \bigcap_{j=1}^N \pi_j^{-1}(I_j)$ , then  $x \in V \subseteq W$ . This shows that the topology generated by  $\mathcal{B}_W$  is also stronger than the weak topology. Taken together with the earlier remark, we see that  $\mathcal{B}_W$  is a countable base for the weak topology, showing that the weakest topology that makes all the projections continuous is second countable.

(c)  $(X, \mathcal{T})$  is a topological space and  $a \in X$ . Define  $\mathcal{T}_a = \{U \in \mathcal{T} | U = \emptyset \text{ or } a \in U\}$ . Show that  $\mathcal{T}_a$  is a topology. If  $(Y, d)$  is a metric space, and  $f : (X, \mathcal{T}_a) \rightarrow (Y, \mathcal{T}_d)$  is continuous, where  $\mathcal{T}_d$  is the metric topology on  $Y$ , show that  $f$  is a constant.

### Solution

1.  $\emptyset$  is in  $\mathcal{T}_a$  and  $a \in X \in \mathcal{T}$  so  $X \in \mathcal{T}_a$ .
2. If  $U, V \in \mathcal{T}_a$  then  $U, V \in \mathcal{T}$  so that  $U \cap V \in \mathcal{T}$ . If  $a \in U \cap V$ , it follows

from the definition that  $U \cap V \in \mathcal{T}_a$ . If  $a \notin U \cap V$ , it follows that  $a \notin U$  or  $a \notin V$  with  $U, V \in \mathcal{T}_a$ . Therefore, either  $U$  or  $V$  is the emptyset, so that  $U \cap V = \emptyset \in \mathcal{T}_a$ .

3. If  $O_\alpha$  is an arbitrary collection of sets in  $\mathcal{T}_a$ , it follows that each of the sets is also in  $\mathcal{T}$ , so that  $\cup_\alpha O_\alpha \in \mathcal{T}$ . If  $a$  is in the union we are done. Otherwise  $a \notin O_\alpha$  for all values of the index  $\alpha \implies \forall \alpha : O_\alpha = \emptyset$ . Then,  $\cup_\alpha O_\alpha = \emptyset \in \mathcal{T}_a$ .

This proves that  $\mathcal{T}_a$  is a topology.

We are given that  $f : (X, \mathcal{T}_a) \rightarrow (Y, \mathcal{T}_d)$  is a continuous function. Let  $U = Y \setminus \{f(a)\}$ . Since  $Y$  is a metric space, it is Hausdorff and the complement of a single point is open. Consequently,  $f^{-1}(U)$  is open in  $\mathcal{T}_a$ , and by construction  $a \notin f^{-1}(U)$ . Therefore,  $f^{-1}(U) = \emptyset \implies f(b) = f(a) \forall b \in X$ , *i.e.*  $f$  is a constant.

(d)  $(X, \mathcal{T})$  is first countable.  $A \subseteq X$ . Show that  $z \in \bar{A}$  the closure of  $A$ , if and only if there is a sequence  $x_n \in A$  such that  $x_n \rightarrow z$ .

You may assume  $\forall x \in X$ , there is a countable collection of nested open sets  $V_1(x) \supseteq V_2(x) \supseteq V_3(x) \supseteq \dots$  that is a local base for the topology.

### Solution

Assume that there is a sequence  $x_n \in A$  such that  $x_n \rightarrow z$ . If  $U \ni z$  is any open set, by convergence of the sequence, there is an index  $N$  such that  $x_n \in U \forall n \geq N$ . In particular  $x_{N+1} \in U \cap A$  showing that  $U \cap A \neq \emptyset$ . Since the open set  $U$  containing  $z$  was arbitrary, it follows that  $z \in \bar{A}$ . Note that this argument holds in any topological space, and we did not use the first countability of  $\mathcal{T}$ .

Conversely, if  $z \in \bar{A}$  and  $\mathcal{T}$  is first countable, here is a countable collection of nested open sets  $V_1(z) \supseteq V_2(z) \supseteq V_3(z) \supseteq \dots$  that is a local base at  $z$ . Consequently  $V_n(z) \cap A$  is non-empty for all  $n$ , and we construct a sequence by picking  $x_n \in V_n(z) \cap A$ . Note that  $x_n \in A$  for all  $n$ , and since the sequence of sets  $V_n(z)$  is nested, it follows that  $x_n \in V_N(z) \forall n \geq N$ . This implies that the sequence  $x_n \rightarrow z$  (Testing convergence with a local base).

Alternatively, if  $U \ni z$  is any open set and the collections of sets  $V_k(z)$  is a local base, there is an index  $N$  such that  $x \in V_N(z) \subseteq U$ . From the above observation, it follows that  $x_n \in U \forall n \geq N$ . Since the set  $U$  was arbitrary, it follows that  $x_n \rightarrow z$ .

(e) The displacement  $u(x, t)$  of a string vibrating in a linear medium is governed by the equation(s)

$$\rho u_{tt} - T u_{xx} = -k u \quad u(0, t) = u(L, t) = 0$$

where  $\rho, T$  and  $k$  are positive constants. Show that the system has a conserved energy  $\mathcal{E} = E[u, u_t]$ , and obtain a bound for the maximum displacement of the string at time  $t$  in terms of the initial data  $u(x, 0)$  and  $u_t(x, 0)$ .

If we multiply both sides by  $u_t$  and integrate over the domain  $[0, L]$  accounting for the boundary conditions and integrating by parts once for the  $u_{xx}u_t$  term, we get

$$\rho \int u_{tt}u_t + T \int u_{xt}u_x + k \int uu_t = 0.$$

The right hand side can be written as time-derivative, and we obtain

$$\frac{1}{2} \frac{d}{dt} \left[ \rho \int u_t^2 + T \int u_x^2 + k \int u^2 \right] = 0.$$

Consequently, if we define the energy by

$$\mathcal{E} = E[u(x, \cdot), u_t(x, \cdot)] = \rho \int u_t^2 + T \int u_x^2 + k \int u^2,$$

it follows that this energy is conserved.

In terms of the initial data, we can define

$$\mathcal{E}_0 = \rho \int_0^L u_t^2(x, 0) dx + T \int_0^L u_x^2(x, 0) dx + k \int_0^L u^2(x, 0) dx$$

In terms of this energy, we have the estimate

$$\int_0^L u_x^2(x, t) dx \leq \frac{\mathcal{E}_0}{T}$$

for all time. A standard argument (spelt out in the homework solutions) using the derivative  $u_x$  to express the function  $u$  in terms of the boundary value  $u(0, t) = 0$  and the fundamental theorem of calculus allows one to conclude  $\|u(\cdot, t)\|_\infty \leq \sqrt{L} \sqrt{\int_0^L u_x^2(x, t) dx}$ . Consequently, the maximum displacement is bounded by  $u_{max} = \sqrt{\mathcal{E}_0 L / T}$ , where  $\mathcal{E}_0$  is defined above.